Computer Projects: Applied Stochastic Analysis

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There are 2 computer projects. The final project reports must be carefully written with \LaTeX{} to include the following points:

- The detailed setup of the problem.
- The procedure you take to do the computation and analysis of the numerical results.
- The issues you encounter and how you overcome.
- Possible discussion about the results and further thinking.

Please submit the hardcopy to our TA. The reports could be composed in either Chinese or English.

1. Potts model.

*Problem.* Apply the Monte Carlo simulations to study the phase transition behavior of the 2D Potts model on the $N \times N$ square lattice with periodic boundary condition. The Hamiltonian of the $q$-state Potts model is defined as

$$
H(\sigma) = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j} - h \sum_i \sigma_i, \quad i = 1, 2, \ldots, N^2
$$

where $\sigma_i = 1, 2, \ldots, q$. Take $q = 3$ or $q = 10$ as concrete example to explore the following problems.

(a) Take $J = 1$, $k_B = 1$ and $h = 0$. Plot the internal energy $u$

$$
u = \frac{U}{N^2} \quad \text{where} \quad U = \langle H \rangle = \frac{1}{N^2} \sum_{\sigma} H(\sigma) \exp(-\beta H(\sigma))
$$

and the specific heat

$$
c = \frac{C}{N^2} \quad \text{where} \quad C = k_B \beta^2 \text{Var}(H)
$$

as the function of temperature $T$, where $\beta = (k_B T)^{-1}$ and $Z = \sum_{\sigma} \exp(-\beta H(\sigma))$ is the partition function. Identify the critical temperature $T_*$ of the phase transition when $N$ is sufficiently large.
(b) For different temperature $T$, plot the magnetization

$$m = \frac{M}{N^2} \quad \text{where} \quad M = \langle \sum_i \sigma_i \rangle$$

as the function of $h$. Can you say something from these plots?

(c) Define the spatial correlation function

$$C(i,j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$

and the correlation length $\xi$ as the characteristic length that $\Gamma(k) = C(i,j)|_{|i-j|=k}$ decays to 0. One can approximate $\Gamma(k)$ by computing the average

$$\Gamma(k) \approx \frac{1}{4N^2} \sum_i \sum_{j \in S_i} C(i,j),$$

where the set

$$S_i = \{ j | i-j = \pm(k,0) \text{ or } \pm(0,k) \},$$

the constant 4 is from four points $j \in S_i$. The correlation length can then be defined through

$$\Gamma(k) \propto \Gamma_0 \exp(-k/\xi), \quad k \gg 1.$$ 

Study the correlation length $\xi$ as the function of $T$ when $h = 0$.

(d) When $h = 0$, investigate the behavior of $c$, $m_+$ and $\xi$ around the critical temperature $T_*$ if we assume the limiting behavior

$$c \sim c_0 \epsilon^{-\gamma} \quad \text{and} \quad \xi \sim \xi_0 \epsilon^{-\delta},$$

where $\epsilon = |1 - T/T_*|$. That is, you need to numerically find the scaling exponents $\gamma$ and $\delta$.

(e) (optional) Study the above problems in the 3D case.


Problem. Numerically solve the following boundary value problem via the simulation of SDEs

$$\begin{cases}
  b \cdot \nabla u + \frac{1}{2} \Delta u = f(x,y), & (x,y) \in B_1(0), \\
  u = \frac{1}{2} & \text{on } (x,y) \in S^1,
\end{cases}$$

where $b = (x,y)$, $f(x,y) = x^2 + y^2 + 1$. We have exact solution $u(x,y) = (x^2 + y^2)/2$ for the model problem. Utilize the standard Euler-Maruyama scheme to do the simulation and check the numerical convergence order in time.

Investigate the multi-level Monte Carlo methodology to solve the above exit problem (optional).