Transport in weak dynamic disorder: A unified theory

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For quantum particles, it is well known that static disorder would lead to Anderson localization (AL) while dynamic (evolving) disorder would destroy AL and facilitate the transport. In this article, we study the transport behavior of a quantum particle in weak dynamic disorder. Based on Wigner representation, we obtain the radiative transfer equation (a linear Boltzmann equation) in the weak dynamic disorder limit, which could lead to not only all the existing transport behaviors in the literature but also new transport behaviors (for example, Lévy flight in momentum space). Furthermore, for dimensions greater than one, though we can formally derive the diffusive transport approximation, we argue that this diffusive transport is not physical but the nondiffusive transport should persist forever. This provides a possible resolution for the long-standing puzzle whether diffusive or nondiffusive transport would prevail in the long time limit. Our result would have major implications for the hypertransport of light, matter wave dynamics in disordered media, and directed polymer problems.

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I. INTRODUCTION

In 1958, Philip Anderson argued that static disorder would arrest transport in the context of disordered crystals [1]. This phenomenon, known as Anderson localization (AL), is very general and has been observed in a variety of systems [2–8]. When the disorder is dynamic (evolving) rather than static in time, it is widely accepted that AL would be destroyed [1,9]. Notable recent topics on this effect include quantum Brownian motor [10], energy transfer in biological complexes [11,12], and quantum walk in disordered environments [13–15].

Despite its fundamental importance, the kind of transport behavior that would be induced by dynamic disorder in the long time limit is still highly debated. It is only agreed that when the disorder is uncorrelated in time, the mean-square displacements in momentum $\langle|\mathbf{p}|^2\rangle$ and in position $\langle|\mathbf{x}|^2\rangle$, respectively, grow with exponents 1 and 3 in time [16,17]. For the disorder with correlation in time in real physical systems, however, diverse transport behaviors have been predicted for the past 20 years [18–22]. Studies based on Newton's equation showed that $\langle|\mathbf{p}|^2\rangle \sim t^{2/5}$ in dimensions greater than one [18,19] while different nondiffusive universal classes would appear in one dimension [20,21]. Meanwhile, a speculative argument suggested that these nondiffusive transport behaviors would transit to the diffusive transport ($\langle|\mathbf{x}|^2\rangle \sim t$) in the long time limit [22]. Notice that these results are based on continuous models which essentially differ from tight-binding models [23–25].

Experimentally, photonic lattice systems and ultracold atom systems have been proved to be promising candidates to address issues related to quantum transport in disordered media [26,27]. More recently, the hypertransport—transport at a rate faster than ballistic—of an optical beam was observed in a two-dimensional photonic lattice system with highly controllable dynamic disorder [28]. However, the fundamental issue for how long this hypertransport can persist remains largely unexplored [20,21].

In this article, we aim to develop a unified theory to address the transport behavior of a quantum particle in weak dynamic disorder. In contrast to the previous works, all of which are based on the Fokker-Planck approximation of Newton's equation [18–22], our theory utilizes the Wigner representation of Schrödinger equation—a first-principle description [29,30]. We derive the radiative transfer approximation (a linear Boltzmann equation) of the Wigner representation at the weak disorder limit. This Boltzmann equation can lead to not only all existing transport behaviors in the literature [18–21] but also new transport behaviors. For instance, we demonstrate that for special dynamic disorder, Lévy flight in momentum space is possible in intermediate time scales. Furthermore, when the evolving velocity of the particle is much faster than that of the disorder, we formally derive the diffusive transport behavior in dimensions greater than one. Nevertheless, we argue that this diffusive transport is not physical but the behavior $\langle|\mathbf{p}|^2\rangle \sim t^{2/5}$ should persist in the long time limit in dimensions greater than one. Our study resolves the debate regarding whether the transport behavior in the long time limit is diffusive or nondiffusive and provides insights for understanding the hypertransport phenomenon observed in Ref. [28].

The article is organized as follows. In Sec. II, the model and assumptions will be presented. The main results which include different asymptotic behaviors will be presented and explained in Sec. III. The detailed derivation of the main results will be deferred in Sec. IV. In Sec. V, we utilize the main results to explain the hypertransport phenomenon of light observed in the experiment [28]. We then conclude the article with a summary in Sec. VI.

II. MODEL AND ASSUMPTIONS

We consider a d-dimensional ($d \geqslant 1$) spatially continuous Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla_{\mathbf{x}}^2\psi + V_0 u(\mathbf{x},t)\psi, \qquad (1)$$

where $V_0 u(\mathbf{x},t)$ is the dynamic disorder and V_0 is the amplitude of the disorder. The power spectrum density $S(\mathbf{k}_r,\omega)$ of $u(\mathbf{x},t)$

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is assumed to be isotropic with respect to $\mathbf{k}_r$. We also assume that there exist finite k_0 and v_0 such that

$$S(\mathbf{k}_r,\omega) = 0 \quad \text{if} \quad |\mathbf{k}_r| > k_0 \quad \text{or} \quad \omega/|\mathbf{k}_r| \gg v_0 \tag{2}$$

and

$$S(\mathbf{k}_r,\omega) \sim S(\mathbf{k}_r,0) \quad \text{if} \quad \omega/|\mathbf{k}_r| \ll v_0. \tag{3}$$

Here k_0 and v_0 can be respectively regarded as the characteristic wave number and phase velocity of the disorder. In essence, this assumption means that both the spectrum and the largest phase velocity of the disorder are finite, which should be satisfied in most of real physical systems (for example, the system studied in Ref. [28]). For simplicity, we also assume that the initial wave packet has a narrow bandwidth and respectively denote the typical wave number, wave vector, angular frequency, and phase velocity as k, $\mathbf{k}$, $\omega(k)$ (equal to $\hbar k^2/2m$) and $v(k)$ (equal to $\hbar k/2m$). The disorder is assumed to be weak. That is, $V_0 \ll \hbar\omega(k)$. We also denote $V_0/2\hbar\omega(k)$ as $\epsilon^{1/2}$.

III. MAIN RESULTS

In this section, we present the main results. In Sec. III A, we first introduce some properties of Wigner function, which is the main tool to explore the different asymptotic behaviors in this article. Then the radiative transfer regime, the Fokker-Planck regime, and the discussion of the existence of diffusive transport regime will be addressed in Secs. III B, III C, and III D, respectively.

A. Wigner function

We first introduce Wigner function $W(\mathbf{x},\mathbf{p},t)$ with respect to $\psi(\mathbf{x},t)$ [29], which is defined as follows:

$$\int \psi\left(\mathbf{x} - \frac{\hbar\mathbf{y}}{2},t\right)\overline{\psi}\left(\mathbf{x} + \frac{\hbar\mathbf{y}}{2},t\right)e^{i\mathbf{p}\cdot\mathbf{y}}\frac{d\mathbf{y}}{(2\pi)^d}.$$

For Wigner function $W(\mathbf{x},\mathbf{p},t)$, the following facts hold:

(1)

$$\int W(\mathbf{x},\mathbf{p},t)|\mathbf{x}|^2 d\mathbf{x}d\mathbf{p} = \int |\mathbf{x}|^2|\psi|^2 d\mathbf{x}.$$

Therefore, the mean-square displacement in position $\langle|\mathbf{x}|^2\rangle$ equals the ensemble average of $\int W(\mathbf{x},\mathbf{p},t)|\mathbf{x}|^2 d\mathbf{x}d\mathbf{p}$.

(2)

$$\int W(\mathbf{x},\mathbf{p},t)|\mathbf{p}|^2 d\mathbf{x}d\mathbf{p} = \hbar^2 \int |\nabla_{\mathbf{x}}\psi|^2 d\mathbf{x}.$$

Therefore, the mean-square displacement in momentum $\langle|\mathbf{p}|^2\rangle$ equals the ensemble average of $\int W(\mathbf{x},\mathbf{p},t)|\mathbf{p}|^2 d\mathbf{x}d\mathbf{p}$.

(3) More importantly, $W(\mathbf{x},\mathbf{p},t)$ satisfies the following random Wigner equation:

$$\frac{\partial W}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W = -iV_0\int \hat{u}\left(\frac{\mathbf{p}_r}{\hbar},\omega\right)\cdot\left[W\left(\mathbf{p} - \frac{\mathbf{p}_r}{2}\right) - W\left(\mathbf{p} + \frac{\mathbf{p}_r}{2}\right)\right] \times e^{i(\frac{\mathbf{p}_r\cdot\mathbf{x}}{\hbar}-\omega t)}\frac{d\mathbf{p}_r d\omega}{(2\pi\hbar)^{d+1}}, \tag{4}$$

where $\hat{u}(\mathbf{k}_r,\omega)$ is the Fourier transform of $u(\mathbf{x},t)$, defined as $\int u(\mathbf{x},t)e^{-i(\mathbf{k}_r\cdot\mathbf{x}-\omega t)}d\mathbf{x}dt$.

Readers should refer to Ref. [30] for detailed derivations of these facts.

B. Radiative transfer regime

In the weak disorder limit $\epsilon \to 0$, we found that the ensemble average of $\epsilon^{-d}W(\mathbf{x}/\epsilon,\mathbf{p},t/\epsilon)$ can be approximated by $W_R(\mathbf{x},\mathbf{p},t)$, which satisfies the following radiative transfer equation:

$$\frac{\partial W_R}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_R = 4\omega^2(k)\int S\left(\frac{\mathbf{p}'-\mathbf{p}}{\hbar},v(k)\frac{|\mathbf{p}'|^2-|\mathbf{p}|^2}{\hbar^2 k}\right) \times [W_R(\mathbf{p}') - W_R(\mathbf{p})]\frac{d\mathbf{p}'}{(2\pi\hbar)^d}. \tag{5}$$

The detailed derivation of this equation can be seen in Sec. IV A. We note that this result is enlightened by Ref. [30] and the corresponding dimensionless form exists in the mathematical literature. In Eq. (5), the quantity

$$S\left(\frac{\mathbf{p}'-\mathbf{p}}{\hbar},v(k)\frac{|\mathbf{p}'|^2-|\mathbf{p}|^2}{\hbar^2 k}\right) \tag{6}$$

represents the scattering intensity between the spectral component of the particle with momentum $\mathbf{p}$ and that of the disorder with momentum $\mathbf{p}'-\mathbf{p}$. According to different ratios of k to k_0 as well as $\max\{v(k),v(k_0)\}$ to v_0, we could obtain different asymptotic regimes, (i), (ii), (iii), and (iv), as illustrated in Fig. 1 and discussed below. For simplicity, we denote $\max\{v(k),v(k_0)\}$ as $v(k)\vee v(k_0)$.

(i) When $k \lesssim k_0$ and $v(k)\vee v(k_0) \ll v_0$ (fast evolving disorder), for $\mathbf{p}$ equal to $\hbar\mathbf{k}$ and $|\mathbf{p}'-\mathbf{p}|$ less than $\hbar k_0$,

$$\frac{v(k)||\mathbf{p}'|^2 - |\mathbf{p}|^2|/\hbar k}{|\mathbf{p}'-\mathbf{p}|} \lesssim 3(v(k)\vee v(k_0)) \ll v_0.$$

According to assumption (3), the scattering intensity (6) can be approximated by $S((\mathbf{p}'-\mathbf{p})/\hbar,0)$. That is, there is no preferred direction for the scattering between the particle and the disorder [see Fig. 1(i)]. Especially when $S(\mathbf{k}_r,0)$ has the following special form:

$$S(\mathbf{k}_r,0) \sim 1/|\mathbf{k}_r|^{d+\theta}, \theta \in (1,2),$$

Lévy flight of the particle in momentum space arises, i.e., $\langle|\mathbf{p}|^2\rangle \sim t^{2/\theta}$ [31]. Similarly to the finite-size effect in Ref. [32], this Lévy flight can only persist for finite time due to finite k_0. For general $S(\mathbf{k}_r,\omega)$, the following relation:

$$d\langle|\mathbf{p} - \hbar\mathbf{k}|^2\rangle/dt > 0 \tag{7}$$

holds. The detailed derivation can be seen in Sec. IV A.

(ii) When $k \lesssim k_0$ and $v(k)\vee v(k_0) \gtrsim v_0$ (slow evolving disorder), the stochastic acceleration of the particle would become slower, as explained below. Under assumption (2), the intensity (6) would be nonzero only when

$$\frac{v(k)||\mathbf{p}'|^2 - |\mathbf{p}|^2|/\hbar k}{|\mathbf{p}'-\mathbf{p}|} \lesssim v_0.$$

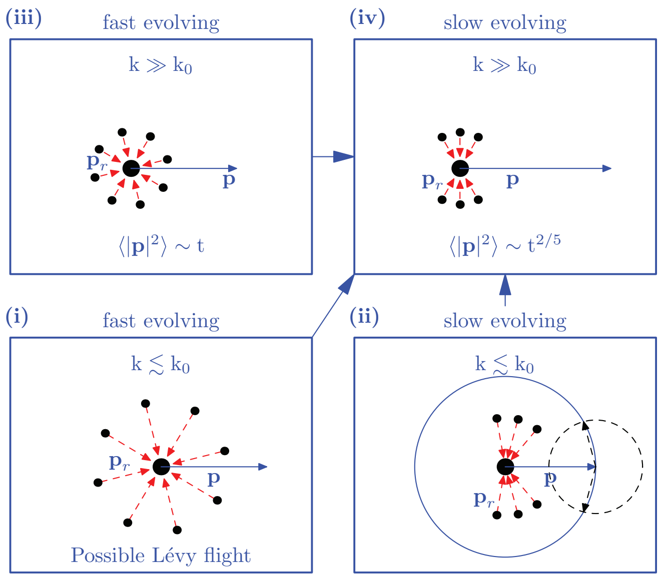

FIG. 1. (Color online) Illustration of different transport behaviors of a quantum particle in weak dynamic (evolving) disorder in dimensions greater than 1. Here k_0 is the characteristic wave number of the disorder while k is the typical wave number of the particle. The smaller black dots are spectral components of the disorder with momenta $\mathbf{p}_r$ ($|\mathbf{p}_r| \leqslant \hbar k_0$) while the larger black dot is the spectral component of the particle with momentum $\mathbf{p}$ ($|\mathbf{p}| = \hbar k$). The fast and slow evolving disorders respectively correspond to the scenario that $\max\{v(k),v(k_0)\} \ll v_0$ and $\max\{v(k),v(k_0)\} \gtrsim v_0$, where v_0 is the characteristic phase velocity of the disorder while $v(k)$ is the typical phase velocity of the particle. In the long time limit, regimes (i), (ii), and (iii) would transit to regime (iv).

Therefore, when $v(k) \gtrsim v_0$, for $\mathbf{p} = \hbar\mathbf{k}$ and fixed $|\mathbf{p}' - \mathbf{p}|$, the particle prefers to be scattered into those $\mathbf{p}'$ satisfying $|\mathbf{p}'| = |\mathbf{p}|$ [see Fig. 1(ii)], rendering the acceleration slower. In this regime, the relation (7) still holds.

C. Fokker-Planck regime

When $k \gg k_0$, $W_R(\mathbf{x},\mathbf{p},t)$ can be further approximated by $W_F(\mathbf{x},\mathbf{p},t)$ satisfying the following equation:

$$\frac{\partial W_F}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} W_F = 2\omega^2(k)\nabla_{\mathbf{p}} \cdot \left(\int S\left(\frac{\mathbf{p}_r}{\hbar}, 2v(k)\frac{\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right) \mathbf{p}_r \otimes \mathbf{p}_r \frac{d\mathbf{p}_r}{(2\pi\hbar)^d} \right) \times \nabla_{\mathbf{p}} W_F(\mathbf{x},\mathbf{p},t). \tag{8}$$

The detailed derivation can be seen in Sec. IV B. We note that this equation coincides with the Fokker-Planck equation derived in Refs. [20,21], which can be connected with the famous Chirikov resonance theory [33]. The quantity

$$S\left(\frac{\mathbf{p}_r}{\hbar}, 2v(k)\frac{\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right) \tag{9}$$

represents the resonance intensity between the spectral component of the particle with momentum $\mathbf{p}$ and that of the disorder with momentum $\mathbf{p}_r$. Based on Eq. (8), we found regimes (iii) and (iv).

(iii) When $k \gg k_0$ and $v(k) \ll v_0$ (fast evolving disorder), $\langle|\mathbf{p}|^2\rangle \sim t$ and $\langle|\mathbf{x}|^2\rangle \sim t^3$, similar to that discussed in Ref. [34]. This can be explained by the following argument. For $\mathbf{p}$ equal to $\hbar\mathbf{k}$,

$$\frac{2v(k)|\mathbf{p}\cdot\mathbf{p}_r|/\hbar k}{|\mathbf{p}_r|} \ll v_0.$$

Therefore, under assumption (3), the resonance density (9) can be approximated by $S(\mathbf{p}_r/\hbar,0)$. That is, there is no preferred direction for the resonance between the particle and the disorder [see Fig. 1(iii)] and the diffusion tensor of Eq. (8) is independent of $\mathbf{p}$, resulting in the hypertransport behavior $\langle|\mathbf{p}|^2\rangle \sim t$ and $\langle|\mathbf{x}|^2\rangle \sim t^3$.

(iv) When $k \gg k_0$ and $v(k) \gtrsim v_0$ (slow evolving disorder), the stochastic acceleration would slow down, which can be explained below. Under assumption (2), the resonance density (9) is nonzero only when

$$\frac{2v(k)|\mathbf{p}\cdot\mathbf{p}_r|/\hbar k}{|\mathbf{p}_r|} \lesssim v_0. \tag{10}$$

Therefore, when $v(k) \gtrsim v_0$, for $\mathbf{p} = \hbar\mathbf{k}$, the particle tends to be scattered by those spectral components of the disorder with momenta $\mathbf{p}_r$ perpendicular to $\mathbf{p}$ rather than those with momenta $\mathbf{p}_r$ parallel to $\mathbf{p}$ [see Fig. 1(iv)]. This renders the stochastic acceleration of the particle slower. In this context, as derived in Ref. [21], $\langle|\mathbf{p}|^2\rangle \sim t^{2/5}$ in dimensions greater than one while $\langle|\mathbf{p}|^2\rangle$ saturates in one dimension under assumption (2).

Now let us address the relation between these behaviors. As shown above, for all dimensions, $\langle|\mathbf{p} - \hbar\mathbf{k}|^2\rangle$ for regimes (i), (ii), and (iii) would grow in time. This means that with time increasing, there are more and more spectral components of the particle with large momentum being excited. Notice that both $v(k)$ and k monotonically increase with respect to k. Therefore, $v(k)$ and k would respectively become much larger than v_0 and k_0 in the long time scale, resulting in the transition from regimes (i), (ii), and (iii) to regime (iv) (see Fig. 1). We note that the transition from regime (iii) to regime (iv) has been found in Ref. [22].

D. Is diffusive transport possible?

In Ref. [22], it is further claimed that the nondiffusive transport behavior $\langle|\mathbf{p}|^2\rangle \sim t^{2/5}$ might transit to the diffusive transport behavior $\langle|\mathbf{x}|^2\rangle \sim t$ in the long time limit. Now we will check this claim for dimensions greater than one. With the formulation of Fokker-Plank equation (8), we provide a formal way to obtain the diffusive transport behavior as below. For $\mathbf{p} = \hbar\mathbf{k} \to \infty$ and fixed $|\mathbf{p}_r|$, according to the condition (10), the resonance intensity (9) would concentrate to those $\mathbf{p}_r$ which satisfy

$$\mathbf{p}\cdot\mathbf{p}_r = 0.$$

That is, under the effect of the disorder, the spectral component of the particle with momentum $\mathbf{p}$ would diffuse on the sphere of radius $\hbar k$ in momentum space. Mathematically, this means that intensity (9) can be approximated by

$$\delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right) S_0\left(\frac{\mathbf{p}_r}{\hbar}\right),$$

where $S_0(\mathbf{p}_r/\hbar)$ is defined by $\int S(\mathbf{p}_r/\hbar,\omega)d\omega$. In this context, W_F would be approximated by W_D, satisfying the following equation:

$$\frac{\partial W_D}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_D = 2\omega^2(k)\nabla_{\mathbf{p}}\cdot\left(\int\delta\left(\frac{2\upsilon(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times\mathbf{p}_r\otimes\mathbf{p}_r\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\right)\nabla_{\mathbf{p}}W_D(\mathbf{x},\mathbf{p},t). \quad (11)$$

It is easy to check that $\langle|\mathbf{p}|^2\rangle$ for Eq. (11) saturates (see the derivation in Sec. IV C). Consequently, diffusive transport would prevail in dimensions greater than one (see the detailed derivation in Sec. IV C).

Notice that the derived diffusive transport is based on the assumption that for fixed $\mathbf{p}$, the resonance intensity concentrates on those $\mathbf{p}_r$ perpendicular to $\mathbf{p}$. However, according to resonance intensity (9), which is related to the Chirikov resonance theory [20,33], for any finite $\mathbf{p}$, there are still amounts of $\mathbf{p}_r$ which are not perpendicular to $\mathbf{p}$ but resonate with $\mathbf{p}$ [see Fig. 1 (iv)]. These $\mathbf{p}_r$ would lead to the increase of $\langle|\mathbf{p}-\hbar\mathbf{k}|^2\rangle$. Furthermore, as shown in Ref. [21], $\langle|\mathbf{p}|^2\rangle\sim t^{2/5}$. Therefore, we believe that the diffusive transport is not physical and $\langle|\mathbf{p}|^2\rangle\sim t^{2/5}$ would persist forever in dimensions greater than one. We speculate that it is also highly probable that diffusive transport would not arise in one dimension, as there is no perpendicular components to alter the direction of $\mathbf{p}$.

IV. DERIVATION OF THE MAIN RESULTS

Here we present the derivations of the main results in Sec. III. The derivations related to the radiative transfer regime, the Fokker-Planck regime, and the diffusive transport regime respectively correspond to Sec. IV A, IV B, and IV C.

A. Radiative transfer regime

In this subsection, we will derive radiative transfer equation (5) as well as relation (7) for both regime (i) and regime (ii).

1. Derivation of radiative transfer equation (5)

To make the derivation more transparent, we divide it into three steps. In the first step, we will transform equation (4) into a dimensionless form. Based on the dimensionless form, we derive the dimensionless form of Eq. (5) in the second step. In the third step, we define $W_R(\mathbf{x},\mathbf{p},t)$ and prove that $W_R(\mathbf{x},\mathbf{p},t)$ satisfies equation (5).

(1) *Step 1:* We note that $\epsilon=(V_0/2\hbar\omega(k))^2$. Then we make the following transformation:

$$\mathbf{x}\to\mathbf{x}/k\epsilon,\quad t\to t/2\omega(k)\epsilon,\quad \psi\to(k\epsilon)^{d/2}\psi^\epsilon.$$

The Wigner function $W^\epsilon(\mathbf{x},\mathbf{p},t)$ corresponding to $\psi^\epsilon(\mathbf{x},t)$ is defined as

$$\int\psi^\epsilon\left(\mathbf{x}-\frac{\epsilon\mathbf{y}}{2},t\right)\overline{\psi^\epsilon}\left(\mathbf{x}+\frac{\epsilon\mathbf{y}}{2},t\right)e^{i\mathbf{p}\cdot\mathbf{y}}\frac{d\mathbf{y}}{(2\pi)^d}.$$

According to the relation between ψ and ψ^ϵ, the following relation holds:

$$W(\mathbf{x},\mathbf{p},t)=(\epsilon/\hbar)^d W^\epsilon(k\epsilon\mathbf{x},\mathbf{p}/\hbar k,2\omega(k)\epsilon t). \quad (12)$$

Similar to $W(\mathbf{x},\mathbf{p},t)$, $W^\epsilon(\mathbf{x},\mathbf{p},t)$ satisfies the following dimensionless Wigner equation,

$$\frac{\partial W^\epsilon}{\partial t}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W^\epsilon = \frac{-i}{\sqrt{\epsilon}}\int\hat\rho(\mathbf{p}_r,\omega)\left[W^\epsilon\left(\mathbf{p}-\frac{\mathbf{p}_r}{2}\right)-W^\epsilon\left(\mathbf{p}+\frac{\mathbf{p}_r}{2}\right)\right]\times e^{i(\mathbf{p}_r\cdot\mathbf{z}-\omega\tau)}\frac{d\mathbf{p}_r d\omega}{(2\pi)^{d+1}}, \quad (13)$$

where $\mathbf{z}\triangleq\mathbf{x}/\epsilon,\tau\triangleq t/\epsilon$, and $\hat\rho$ is the Fourier transform of $\rho(\mathbf{x},t)$ (defined by $u(\mathbf{x}/k,t/2\omega(k))$). Denote the power spectrum density of $\rho(\mathbf{x},t)$ as $S_\rho(\mathbf{k}_r,\omega)$. It is easy to find that $S_\rho(\mathbf{k}_r,\omega)=2k^d\omega(k)S(k\mathbf{k}_r,2\omega(k)\omega)$.

(2) *Step 2:* Now we come to address the asymptotic behavior of W^ϵ. We use the asymptotic techniques developed in Ref. [30]. Assume that W^ϵ has the following expansion:

$$W^\epsilon(\mathbf{x},\mathbf{p},t)=W_0(\mathbf{x},\mathbf{p},t)+\epsilon^{1/2}W_1(\mathbf{x},\mathbf{z},\mathbf{p},t,\tau)+\epsilon W_2(\mathbf{x},\mathbf{z},\mathbf{p},t,\tau)+\cdots. \quad (14)$$

Substituting expansion (14) into Eq. (13), we obtain the following equations:

$$\text{For } O\,(\epsilon^{-\frac{1}{2}}):\ \frac{\partial W_1}{\partial\tau}+\mathbf{p}\cdot\nabla_{\mathbf{z}}W_1+\theta W_1 = -i\int\hat\rho(\mathbf{p}_r,\omega)\left[W_0\left(\mathbf{p}-\frac{\mathbf{p}_r}{2}\right)-W_0\left(\mathbf{p}+\frac{\mathbf{p}_r}{2}\right)\right]\times e^{i(\mathbf{p}_r\cdot\mathbf{z}-\omega\tau)}\frac{d\mathbf{p}_r d\omega}{(2\pi)^{d+1}}. \quad (15)$$

$$\text{For } O\,(1):\ \frac{\partial W_0}{\partial t}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W_0+\frac{\partial W_2}{\partial\tau}+\mathbf{p}\cdot\nabla_{\mathbf{z}}W_2 = -i\int\hat\rho(\mathbf{p}_r,\omega)\left[W_1\left(\mathbf{p}-\frac{\mathbf{p}_r}{2}\right)-W_1\left(\mathbf{p}+\frac{\mathbf{p}_r}{2}\right)\right]\times e^{i(\mathbf{z}\cdot\mathbf{p}_r-\omega\tau)}\frac{d\mathbf{p}_r d\omega}{(2\pi)^{d+1}}. \quad (16)$$

In Eq. (15), we have added the term θW_1, which will be eliminated by letting $\theta\to 0$ in the end. According to Eq. (15), $W_1(\mathbf{x},\mathbf{z},\mathbf{p},t,\tau)$ can be expressed as

$$\int\tilde W_1(\mathbf{x},\mathbf{p}_r,\mathbf{p},t,\omega)e^{i(\mathbf{p}_r\cdot\mathbf{z}-\omega\tau)}\frac{d\mathbf{p}_r d\omega}{(2\pi)^{d+1}}, \quad (17)$$

where

$$\tilde W_1(\mathbf{x},\mathbf{p}_r,\mathbf{p},t,\omega)\triangleq\frac{-i}{i\omega+i\mathbf{p}\cdot\mathbf{p}_r+\theta}\cdot\hat\rho(\mathbf{p}_r,\omega)\times\left[W_0\left(\mathbf{p}-\frac{\mathbf{p}_r}{2}\right)-W_0\left(\mathbf{p}+\frac{\mathbf{p}_r}{2}\right)\right].$$

For simplicity, denote the ensemble average as $\mathbb{E}$. Taking the ensemble average for Eq. (16) and assuming $\mathbb{E}\{\frac{\partial W_2}{\partial\tau}+\mathbf{p}\cdot\nabla_{\mathbf{z}}W_2\}=0$, we can get

$$\frac{\partial W_0}{\partial t}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W_0=\mathcal{L}_R W_0, \quad (18)$$

where

$$\mathcal{L}_R W_0 \triangleq -i\mathbb{E}\left\{\int \hat{\rho}(\mathbf{p}_r,\omega)\left[W_1\left(\mathbf{p}-\frac{\mathbf{p}_r}{2}\right)-W_1\left(\mathbf{p}+\frac{\mathbf{p}_r}{2}\right)\right] \times e^{i(\mathbf{z}\cdot\mathbf{p}_r-\omega\tau)}\frac{d\mathbf{p}_r d\omega}{(2\pi)^{d+1}}\right\}. \tag{19}$$

Inserting the expression (17) of W_1 into Eq. (19) and using the fact that

$$\mathbb{E}\{\hat{\rho}(\mathbf{p}_1,\omega_1)\hat{\rho}(\mathbf{p}_2,\omega_2)\} = S_\rho(\mathbf{p}_1,\omega_1)\delta_{\mathbf{p}_1+\mathbf{p}_2}\delta_{\omega_1+\omega_2},$$

we can get

$$\mathcal{L}_R W_0 = \int S_\rho(\mathbf{p}_r,\omega)\left\{\frac{\theta}{\theta^2+[(\mathbf{p}+\mathbf{p}_r/2)\cdot\mathbf{p}_r-\omega]^2} + \frac{\theta}{\theta^2+[(\mathbf{p}+\eta\mathbf{p}_r/2)\cdot\mathbf{p}_r+\omega]^2}\right\} \times [W_0(\mathbf{p}+\mathbf{p}_r)-W_0(\mathbf{p})]\frac{d\mathbf{p}_r d\omega}{(2\pi)^{d+1}}.$$

Using the fact that

$$\frac{1}{\pi}\frac{\theta}{x^2+\theta^2} \to \delta(x), \quad \text{as} \quad \theta \to 0+,$$

where $\delta(x)$ is the Dirac δ function, we can obtain

$$\mathcal{L}_R W_0 = \int S_\rho\left(\mathbf{p}_r,\left(\mathbf{p}+\frac{\mathbf{p}_r}{2}\right)\cdot\mathbf{p}_r\right)[W_0(\mathbf{p}+\mathbf{p}_r)-W_0(\mathbf{p})]\frac{d\mathbf{p}_r}{(2\pi)^d} = \int S_\rho\left(\mathbf{p}'-\mathbf{p},\frac{|\mathbf{p}'|^2-|\mathbf{p}|^2}{2}\right)[W_0(\mathbf{p}')-W_0(\mathbf{p})]\frac{d\mathbf{p}'}{(2\pi)^d}.$$

By combining Eq. (18) with this expression, we obtain the dimensionless form of Eq. (5).

(3) *Step 3:* We introduce $W_R(\mathbf{x},\mathbf{p},t)$ as follows:

$$W_R(\mathbf{x},\mathbf{p},t) \triangleq \hbar^{-d} W_0(k\mathbf{x},\mathbf{p}/\hbar k,2\omega(k)t). \tag{20}$$

When $V_0/\hbar\omega(k) \ll 1$, since $\mathbb{E}W^\epsilon \to W_0$ as $\epsilon \to 0$, according to relation (12),

$$\epsilon^{-d}\mathbb{E}W(\mathbf{x}/\epsilon,\mathbf{p},t/\epsilon) \to W_R(\mathbf{x},\mathbf{p},t) \tag{21}$$

as $\epsilon \to 0$. Now let us check whether $W_R(\mathbf{x},\mathbf{p},t)$ satisfies Eq. (5). According to the definition of $W_R(\mathbf{x},\mathbf{p},t)$, the following relations hold:

$$\hbar^d\frac{\partial W_R(\mathbf{x},\mathbf{p},t)}{\partial t} = 2\omega(k)\frac{\partial W_0(k\mathbf{x},\mathbf{p}/\hbar k,2\omega(k)t)}{\partial t},$$

$$\hbar^d\frac{\partial W_R(\mathbf{x},\mathbf{p},t)}{\partial x_i} = k\frac{\partial W_0(k\mathbf{x},\mathbf{p}/\hbar k,2\omega(k)t)}{\partial x_i}.$$

Using the facts that

$$(\mathcal{L}_R W_0)(\mathbf{p}/\hbar k) = (\hbar k)^{-d}\int S_\rho\left(\frac{\mathbf{p}'-\mathbf{p}}{\hbar k},\frac{|\mathbf{p}'|^2-|\mathbf{p}|^2}{2\hbar^2 k^2}\right) \times [W_R(\mathbf{p}')-W_R(\mathbf{p})]\frac{d\mathbf{p}'}{(2\pi)^d}$$

and

$$S_\rho(\mathbf{k}_r,\omega) = 2k^d\omega(k)S(k\mathbf{k}_r,2\omega(k)\omega),$$

we get

$$(\mathcal{L}_R W_0)(\mathbf{p}/\hbar k) = 2\omega(k)\int S\left(\frac{\mathbf{p}'-\mathbf{p}}{\hbar},v(k)\frac{|\mathbf{p}'|^2-|\mathbf{p}|^2}{\hbar^2 k}\right) \times [W_R(\mathbf{p}')-W_R(\mathbf{p})]\frac{d\mathbf{p}'}{(2\pi\hbar)^d}.$$

These facts lead to Eq. (5), which completes the derivation.

2. *Derivation of relation (7) for regime (i) and regime (ii)*

To derive relation (7), we introduce the following assumption: $W_R(\mathbf{x},\hbar\mathbf{k}+\mathbf{p},t)$ is isotropic with respect to $\mathbf{p}$ and monotonically decreases with respect to $|\mathbf{p}|$. Then we can prove

$$d\langle|\mathbf{p}-\hbar\mathbf{k}|^2\rangle/dt > 0 \tag{22}$$

for regime (i).

Proof.

$$\begin{aligned}\frac{d}{dt}\langle|\mathbf{p}-\hbar\mathbf{k}|^2\rangle &= \frac{d}{dt}\int|\mathbf{p}-\hbar\mathbf{k}|^2 W(\mathbf{x},\mathbf{p},t)d\mathbf{x}d\mathbf{p}\\ &\sim \frac{d}{dt}\int|\mathbf{p}-\hbar\mathbf{k}|^2 W_R(\mathbf{p},t)d\mathbf{p}\\ &= 4\omega^2(k)\int\int|\mathbf{p}-\hbar\mathbf{k}|^2 S\left(\frac{\mathbf{p}'-\mathbf{p}}{\hbar},0\right) \times [W_R(\mathbf{p}')-W_R(\mathbf{p})]\frac{d\mathbf{p}'d\mathbf{p}}{(2\pi\hbar)^d}\\ &= 4\omega^2(k)\int\int|\mathbf{p}'-\hbar\mathbf{k}|^2 S\left(\frac{\mathbf{p}-\mathbf{p}'}{\hbar},0\right) \times [W_R(\mathbf{p})-W_R(\mathbf{p}')]\frac{d\mathbf{p}d\mathbf{p}'}{(2\pi\hbar)^d}\\ &= 2\omega^2(k)\int\int(|\mathbf{p}-\hbar\mathbf{k}|^2-|\mathbf{p}'-\hbar\mathbf{k}|^2) \times S\left(\frac{\mathbf{p}'-\mathbf{p}}{\hbar},0\right)[W_R(\mathbf{p}')-W_R(\mathbf{p})]\frac{d\mathbf{p}'d\mathbf{p}}{(2\pi\hbar)^d}\\ &> 0.\end{aligned}$$

Here, we have used the assumption to obtain the last inequality. ■

Similarly, we can prove relation (22) for regime (ii).

B. Fokker-Planck regime

In this subsection, based on radiative transfer equation (5), we derive Fokker-Planck equation (8). We assume $k \gg k_0$ in this Fokker-Planck regime. To simplify the derivation, we transform Eq. (5) into the following form:

$$\frac{\partial W_R}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_R = 4\omega^2(k)\int S\left(\frac{\mathbf{p}_r}{\hbar},v(k)\cdot\frac{|\mathbf{p}+\mathbf{p}_r|^2-|\mathbf{p}|^2}{\hbar^2 k}\right) \times [W_R(\mathbf{p}+\mathbf{p}_r)-W_R(\mathbf{p})]\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}. \tag{23}$$

Since $|\mathbf{p}| = \hbar k \gg \hbar k_0 \geqslant \hbar|\mathbf{k}_r| = |\mathbf{p}_r|$, by taking the Taylor expansion, we can get that the right-hand term of Eq. (23) can be approximated by

$$4\omega^2(k)\int\left\{\partial_\omega S\left(\frac{\mathbf{p}_r}{\hbar},v(k)\frac{2\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)v(k)\frac{|\mathbf{p}_r|^2}{\hbar^2 k}\mathbf{p}_r\cdot\nabla_{\mathbf{p}}W_F(\mathbf{p})+\sum_{i,j}S\left(\frac{\mathbf{p}_r}{\hbar},v(k)\frac{2\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)\frac{1}{2}p_{ri}p_{rj}\partial_{p_i p_j}W_F(\mathbf{p})\right\}\frac{d\mathbf{p}_r}{(2\pi\hbar)^d},$$

which equals

$$2\omega^2(k)\nabla_{\mathbf{p}}\cdot\left(\int S\left(\frac{\mathbf{p}_r}{\hbar},2v(k)\frac{\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)\times\mathbf{p}_r\otimes\mathbf{p}_r\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\right)\nabla_{\mathbf{p}}W_F(\mathbf{x},\mathbf{p},t).$$

This completes the derivation of Eq. (8).

C. Diffusive transport regime

In this subsection, we address the properties of Eq. (11) for dimensions greater than one. Two properties would be derived. The first is about the saturation of $\langle|\mathbf{p}|^2\rangle$. The second is about the diffusive transport approximation.

1. Derivation of the saturation of $\langle|\mathbf{p}|^2\rangle$

The following relation holds:

$$\frac{d\int|\mathbf{p}|^2W_D(\mathbf{x},\mathbf{p},t)d\mathbf{x}d\mathbf{p}}{dt}\equiv 0.$$

Proof. According to Eq. (11), we obtain

$$\begin{aligned}\frac{d\int|\mathbf{p}|^2W_D(\mathbf{x},\mathbf{p},t)d\mathbf{x}d\mathbf{p}}{dt}&=2\omega^2(k)\int|\mathbf{p}|^2\nabla_{\mathbf{p}}\cdot\left(\int\delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times\mathbf{p}_r\otimes\mathbf{p}_r\frac{d\mathbf{p}_r}{(2\pi)^d}\right)\nabla_{\mathbf{p}}W_D(\mathbf{x},\mathbf{p},t)d\mathbf{x}d\mathbf{p}\\&=-4\omega^2(k)\int\delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times(\mathbf{p}\cdot\mathbf{p}_r)(\mathbf{p}_r\cdot\nabla_{\mathbf{p}}W_D(\mathbf{x},\mathbf{p},t))\frac{d\mathbf{p}_r}{(2\pi)^d}d\mathbf{x}d\mathbf{p}.\end{aligned}$$

Noticing that there exists a term like

$$\delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)(\mathbf{p}\cdot\mathbf{p}_r),$$

we complete the proof. ■

2. Derivation of the diffusive transport approximation in dimensions greater than one

In Sec. IV C1, we have shown that the mean-square displacement in momentum $\langle|\mathbf{p}|^2\rangle$ saturates. Here, we will further demonstrate that diffusive transport will arise when $\langle|\mathbf{p}|^2\rangle$ saturates in dimensions greater than one. To this end, we rescale time and space of equation (11) as follows:

$$t\to t/\xi^2,\quad \mathbf{x}\to\mathbf{x}/\xi,$$

where ξ is a small dimensionless parameter. With this rescaling, we get the following equation:

$$\begin{aligned}&\xi^2\frac{\partial W_D^\xi}{\partial t}+\xi\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_D^\xi\\&=2\omega^2(k)\nabla_{\mathbf{p}}\cdot\left(\int\delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)\times S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\mathbf{p}_r\otimes\mathbf{p}_r\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\right)\nabla_{\mathbf{p}}W_D^\xi(\mathbf{x},\mathbf{p},t).\end{aligned}\tag{24}$$

For simplicity, denote the right-hand side term as $\mathcal{L}_D W_D^\xi(\mathbf{x},\mathbf{p},t)$. Assume that

$$W_D^\xi(\mathbf{x},\mathbf{p},t)=W_{D,0}(\mathbf{x},\mathbf{p},t)+\xi W_{D,1}(\mathbf{x},\mathbf{p},t)+\xi^2W_{D,2}(\mathbf{x},\mathbf{p},t)+\cdots.$$

Inserting this expansion into equation (24), we get the following equations:

$$O(1):\mathcal{L}_D W_{D,0}(\mathbf{p})=0,\tag{25}$$

$$O(\xi):(\mathcal{L}_D W_{D,1})(\mathbf{p})=\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_{D,0},\tag{26}$$

$$O(\xi^2):(\mathcal{L}_D W_{D,2})(\mathbf{p})=\frac{\partial W_{D,0}}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_{D,1}.\tag{27}$$

According to Lemma 1, Eq. (25) would result in the fact that

$$W_{D,0}(\mathbf{p})=W_{D,0}(|\mathbf{p}|).$$

Assume that $W_{D,1}(\mathbf{x},\mathbf{p},t)=C(|\mathbf{p}|)\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_{D,0}$. According to Lemma 2, Eq. (26) becomes

$$-C(|\mathbf{p}|)\lambda(|\mathbf{p}|)\mathbf{p}\cdot\nabla_{\mathbf{x}}W_{D,0}=\mathbf{p}\cdot\nabla_{\mathbf{x}}W_{D,0},$$

where $\lambda(|\mathbf{p}|)$ is given in Eq. (A1). Therefore,

$$W_{D,1}=-\frac{1}{\lambda(|\mathbf{p}|)}\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{x}}W_{D,0}.\tag{28}$$

To ensure Eq. (27) is solvable, according to the Fredholm alternative theorem,

$$\int_{|\mathbf{p}'|=|\mathbf{p}|}\left\{\frac{\partial W_{D,0}}{\partial t}+\frac{\mathbf{p}'}{m}\cdot\nabla_{\mathbf{x}}W_{D,1}\right\}d\Omega(\hat{\mathbf{p}}')=0.$$

Substituting Eq. (28) into the above equation, we can get

$$\int_{|\mathbf{p}'|=|\mathbf{p}|}\left\{\frac{\partial W_{D,0}}{\partial t}-\frac{\mathbf{p}'}{m}\cdot\nabla_{\mathbf{x}}\left(\frac{1}{\lambda(|\mathbf{p}|)}\frac{\mathbf{p}'}{m}\cdot\nabla_{\mathbf{x}}W_{D,0}\right)\right\}d\Omega(\hat{\mathbf{p}}')=0.$$

Notice that

$$\int\hat{p}_i'\hat{p}_j'd\Omega(\hat{\mathbf{p}}')=0,\quad\text{if}\quad i\neq j$$

and that

$$\int(\hat{p}_i')^2d\Omega(\hat{\mathbf{p}}')=\frac{\Gamma_d}{d},\quad\text{for}\quad i=1,\ldots,d,$$

where Γ_d is the surface area of d-dimensional unit sphere. Therefore, we get the following diffusive transport equation:

$$\frac{\partial W_{D,0}}{\partial t}=D(|\mathbf{p}|)\nabla_{\mathbf{x}}^2W_{D,0},\tag{29}$$

where the diffusion coefficient $D(|\mathbf{p}|)$ equals

$$\begin{cases} \pi^2|\mathbf{p}|^3/\left(dm^3\hbar^2\omega^2(k)\int_0^\infty S_0(k)k^2dk\right), & d=2, \\ 2^{d-1}\pi^d|\mathbf{p}|^3/\left(dm^3\hbar^2\omega^2(k)\Gamma_{d-1}\int_0^\infty S_0(k)k^ddk\right), & d\geqslant 3. \end{cases} \tag{30}$$

Here, $S_0(|\mathbf{k}|)$ is defined by $S_0(\mathbf{k})$. Equation (29) exactly means that under the effect of the disorder, the spectral component of the particle with momentum $\mathbf{p}$ would move on the sphere of radius $|\mathbf{p}|$ in momentum space, rendering the transport diffusive.

V. APPLICATIONS

Besides the fundamental significance, our results provide a solid theoretical basis to explain and predict experimental results. In the following, we will explain how they can be applied to photonic lattice systems. In the transverse localization scheme [26,28], a photonic lattice system can be described by

$$\frac{i}{k'}\frac{\partial\psi}{\partial z} = -\frac{1}{2k'^2}\nabla_{\mathbf{x}}^2\psi - \frac{\Delta n(\mathbf{x},z)}{n'}\psi, \tag{31}$$

where z is the propagation coordinate and $\mathbf{x}=(x,y)$ is the transverse coordinate. The wave number k' is $2\pi\omega n'$, where ω is the optical frequency and n' is the average refractive index. ψ is the envelope such that optical field $E(\mathbf{x},z,t) = \psi(\mathbf{x},z)e^{i(k'z-\omega t)}$. The relative random fluctuation of refractive index $-\Delta n/n'$ can be written as $V_0u(k_{\perp 0}\mathbf{x},k_{z0}z)$, where V_0, $k_{\perp 0}$, and k_{z0} are, respectively, the amplitude, the characteristic transverse wave number, and the characteristic beating rate in z of the disorder. In the experiment [28], $k_{z0} = k_{\perp 0}\delta k/k'$, where δk is a highly controllable parameter. When δk varied from zero to a sufficiently large nonzero value, the transition from Anderson localization regime to the hypertransport regime was observed [28].

However, it is natural to wonder how long the hypertransport and momentum spectral expansion can persist for when δk is nonzero? Our theory can provide the answer for this question, provided that the paraxial equation (31) is valid for describing the transport behavior of the light. First, notice that the relevant wave number of the initial wave packet is not k' but the transverse wave number $k_\perp$. Let $1/k' \to \hbar$, $1\to m$ and $z\to t$. Then we recover Eq. (1). In the experiment of Ref. [28], $V_0/\hbar\omega(k_\perp) = V_0k_\perp^2/2k'^2 \ll 1$, which fulfills our weak disorder assumption. Note that $k_{\perp 0}$ and $k_{z0}/k_{\perp 0}$ respectively are the k_0 and v_0 in assumption (2) and (3). In the experiment of Ref. [28], $k_\perp$ is about 0.07 μm^{-1}, which is much smaller than k_0 (about $0.7\mu\text{m}^{-1}$), while $v(k_\perp)$ (equal to $k_\perp/k'$) is the same order as v_0 (equal to $\delta k/k'$). According to the main result, the system is in regime (ii), where spectral expansion takes place. As time increases, the main result tells us the system would enter into regime (iv), where $\langle|\mathbf{p}|^2\rangle \sim t^{2/5}$. As argued above, this hypertransport behavior would persist until the system enters into a regime where paraxial equation description is invalid.

The range of application of our results is not restricted to photonic lattice systems but extends to ultracold atom systems where the time dependence of the disorder could be induced by a modulation of the intensity of the electronic field [6] and to random directed polymer problems which can be described by the imaginary version of Eq. (1) [22,35].

VI. CONCLUSION AND DISCUSSION

In this article, we have developed a unified theory to address the transport behaviors of a quantum particle in weak dynamic disorder by utilizing Wigner representation of the Schrödinger equation. We have derived a linear Boltzmann equation, which could lead to not only all the existing behaviors in the literature but also new transport behaviors. Furthermore, for dimensions greater than one, we have demonstrated that $\langle|\mathbf{p}|^2\rangle \sim t^{2/5}$ should persist forever, which settles the dispute whether diffusive or nondiffusive transport would prevail in the long time limit.

Our results also draw prospects for exploring a variety of issues relevant for transport behaviors in dynamically disordered environments. For instance, it is interesting to incorporate dissipation effect [36]. Moreover, with the growing interest on many-body disordered quantum systems, it is extremely important to study the interplay between interactions and dynamic disorder. Here intriguing issues would arise when the interactions are attractive, as attractive interactions tend to localize the wave packet via self-trapping mechanisms while dynamic disorder would delocalize the wave packet. In this respect, some efforts have been made [37,38] and further studies could be advanced with incorporation of photonic lattice systems [28] and ultracold atom systems [6], where both the dynamic disorder and the interactions can be introduced in a highly controllable way. We note that the Wigner function used here could be measured in experiments via interferometric methods [39], which probably enables characterizing transport behaviors of disordered quantum systems in a new way. Finally, we note that all behaviors studied here belong to the Bragg regime, where one spectral component of the particle is scattered by one spectral component of the disorder at a time. For the Raman-Nath regime, where one spectral component of the particle would be scattered by several spectral components of the disorder simultaneously, studies have begun to appear [40].

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APPENDIX A: PROOF OF LEMMA 1 and 2

Lemma 1.

$$(\mathcal{L}_D f)(|\mathbf{p}|) = 0.$$

Proof.

$$(\mathcal{L}_D f)(|\mathbf{p}|) = 2\omega^2(k)\sum_i \nabla_{p_i}\left(\int \delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right) \times p_{ri}\mathbf{p}_r\cdot\mathbf{p}\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|}\right).$$

We split the right-hand term into the summation of the following three terms:

$$\mathrm{I} = 2\omega^2(k)\cdot\frac{2v(k)}{\hbar^2 k}\int \delta'\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times|\mathbf{p}_r|^2(\mathbf{p}\cdot\mathbf{p}_r)\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|},$$

$$\mathrm{II} = 2\omega^2(k)\int \delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)|\mathbf{p}_r|^2\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|}$$

and

$$\mathrm{III} = 2\omega^2(k)\int \delta\left(\frac{2v(k)\mathbf{p}\cdot\mathbf{p}_r}{\hbar^2 k}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times(\mathbf{p}\cdot\mathbf{p}_r)^2\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\left(\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|}\right)'.$$

$\forall$ fixed $\mathbf{p}$, there exists an orthogonal matrix Q such that $Q\mathbf{p} = (|\mathbf{p}|,0,\ldots,0)^T$. Therefore,

$$\mathrm{I} = 2\omega^2(k)\frac{2v(k)}{\hbar^2 k}\int \delta'\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|p_{r1}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times|\mathbf{p}_r|^2 p_{r1}\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}f'(|\mathbf{p}|),$$

$$\mathrm{II} = 2\omega^2(k)\int \delta\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|p_{r1}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)|\mathbf{p}_r|^2\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|}$$

and

$$\mathrm{III} = 2\omega^2(k)\int \delta\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|p_{r1}\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times|\mathbf{p}|^2 p_{r1}^2\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\left(\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|}\right)'.$$

When $d \geqslant 3$, expanding I, II, and III in the spherical coordinate, we get

$$\mathrm{I} = 2\omega^2(k)\frac{2v(k)}{\hbar^2 k}\frac{\Gamma_{d-1}}{(2\pi\hbar)^d}\int_0^\infty\int_0^\pi S_0\left(\frac{r}{\hbar}\right)r^{d+2}\times\delta'\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|r\cos\theta\right)\cos\theta\sin^{d-2}\theta d\theta dr f'(|\mathbf{p}|),$$

$$\mathrm{II} = 2\omega^2(k)\frac{\Gamma_{d-1}}{(2\pi\hbar)^d}\int_0^\infty\int_0^\pi S_0\left(\frac{r}{\hbar}\right)r^{d+1}\times\delta\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|r\cos\theta\right)\sin^{d-2}\theta d\theta dr\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|},$$

and

$$\mathrm{III} = 2\omega^2(k)\cdot\frac{\Gamma_{d-1}|\mathbf{p}|^2}{(2\pi\hbar)^d}\int_0^\infty\int_0^\pi S_0\left(\frac{r}{\hbar}\right)r^{d+1}\times\delta\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|r\cos\theta\right)\cos^2\theta\sin^{d-2}\theta d\theta dr\left(\frac{f'(|\mathbf{p}|)}{|\mathbf{p}|}\right)'.$$

Here Γ_{d-1} is the surface area of a $(d-1)$-dimensional unit sphere. Using the following facts:

$$\int_0^\pi \delta'(C\cos\theta)\cos\theta\sin^n\theta d\theta = -1/C^2,$$

$$\int_0^\pi \delta(C\cos\theta)\sin^n\theta d\theta = 1/C,$$

and

$$\int_0^\pi \delta(C\cos\theta)\cos^2\theta\sin^n\theta d\theta = 0,$$

we can get

$$\mathrm{I}+\mathrm{II} = 0,\quad \mathrm{III} = 0.$$

Therefore, $(\mathcal{L}_D f)(|\mathbf{p}|) = 0$. When $d = 2$, we can similarly prove

$$\mathrm{I}+\mathrm{II} = 0,\quad \mathrm{III} = 0.$$

Therefore, $(\mathcal{L}_D f)(|\mathbf{p}|) = 0$. ■

Lemma 2.

$$\mathcal{L}_D(\mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|)) = -\lambda(|\mathbf{p}|)\mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|),$$

where

$$\lambda(|\mathbf{p}|) = \begin{cases}\frac{m\hbar^2\omega^2(k)}{\pi^2|\mathbf{p}|^3}\int_0^\infty S_0(k_r)k_r^2 dk_r & d = 2,\\ \frac{2m\hbar^2\omega^2(k)}{|\mathbf{p}|^3}\frac{\Gamma_{d-1}}{(2\pi)^d}\int_0^\infty S_0(k_r)k_r^d dk_r & d\geqslant 3.\end{cases} \tag{A1}$$

Here Γ_{d-1} is the surface area of a $(d-1)$-dimensional unit sphere.

Proof. Notice that

$$\nabla_{p_j}(\mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|)) = f_j(|\mathbf{p}|) + \frac{p_j}{|\mathbf{p}|}\mathbf{p}\cdot\mathbf{f}'(|\mathbf{p}|).$$

Therefore, $\mathcal{L}_D(\mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|))$ can be split into the summation of I and II, where

$$\mathrm{I} = 2\omega^2(k)\sum_i\nabla_{p_i}\left(\int\delta\left(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times p_{ri}\mathbf{p}_r\cdot\mathbf{f}(|\mathbf{p}|)\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\right)$$

and

$$\mathrm{II} = 2\omega^2(k)\sum_i\nabla_{p_i}\left(\int\delta\left(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times p_{ri}\mathbf{p}\cdot\mathbf{p}_r\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}\frac{\mathbf{p}\cdot\mathbf{f}'(|\mathbf{p}|)}{|\mathbf{p}|}\right).$$

Notice that there exists $\delta(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r)\mathbf{p}\cdot\mathbf{p}_r$ in the kernel of the integral of term II. Therefore, $\mathrm{II} = 0$:

$$2\omega^2(k)\cdot\mathrm{I} = \frac{2v(k)}{\hbar^2 k}\int\delta'\left(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r\right)\times S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)|\mathbf{p}_r|^2\mathbf{p}_r\cdot\mathbf{f}(|\mathbf{p}|)\frac{d\mathbf{p}_r}{(2\pi\hbar)^d} + \int\delta\left(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r\right)S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)\times(\mathbf{p}\cdot\mathbf{p}_r)\mathbf{p}_r\cdot\frac{\mathbf{f}'(|\mathbf{p}|)}{|\mathbf{p}|}\frac{d\mathbf{p}_r}{(2\pi\hbar)^d}. \tag{A2}$$

The second term of the right-hand side of Eq. (A2) equals zero, since there exists $\delta(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r)\mathbf{p}\cdot\mathbf{p}_r$ in the kernel of the integral. Now we come to calculate the first term. $\forall$ fixed $\mathbf{p}$, there exists an orthogonal matrix Q such that $Q\mathbf{p} =$

$(|\mathbf{p}|,0,\ldots,0)^T$. Therefore, when $d \geqslant 3$,

$$\begin{aligned} \mathrm{I} &= 2\omega^2(k)\frac{2v(k)}{\hbar^2 k}\frac{\Gamma_{d-1}}{(2\pi\hbar)^d} \\ &\quad\times \int_0^\infty \int_0^\pi \delta'\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|r\cos\theta\right) S_0\left(\frac{r}{\hbar}\right) \\ &\quad\times r^{d+2}\cos\theta \sin^{d-2}\theta d\theta dr (Q\mathbf{f}(|\mathbf{p}|))_1. \end{aligned}$$

Since

$$\int_0^\pi \delta'(C\cos\theta)\cos\theta\sin^n\theta d\theta = -1/C^2,$$

then

$$\mathrm{I} = -\frac{2m\hbar^2\omega^2(k)}{|\mathbf{p}|^2}\frac{\Gamma_{d-1}}{(2\pi)^d}\int_0^\infty S_0(k_r)k_r^d dk_r (Q\mathbf{f}(|\mathbf{p}|))_1.$$

Using the fact that

$$(Q\mathbf{f}(|\mathbf{p}|))_1 = \frac{1}{|\mathbf{p}|}(|\mathbf{p}|,\ldots,0)^T \cdot Q\mathbf{f}(|\mathbf{p}|) = \frac{1}{|\mathbf{p}|}\mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|),$$

we can get

$$\mathrm{I} = -\frac{2m\hbar^2\omega^2(k)}{|\mathbf{p}|^3}\frac{\Gamma_{d-1}}{(2\pi)^d}\int_0^\infty S_0(k_r)k_r^d dk_r \mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|).$$

When $d = 2$,

$$\begin{aligned} \mathrm{I} &= 2\omega^2(k)\frac{2v(k)}{\hbar^2 k}\int \delta'\left(\frac{2v(k)}{\hbar^2 k}\mathbf{p}\cdot\mathbf{p}_r\right) \\ &\quad\times S_0\left(\frac{\mathbf{p}_r}{\hbar}\right)|\mathbf{p}_r|^2\mathbf{p}_r\cdot\mathbf{f}(|\mathbf{p}|)\frac{d\mathbf{p}_r}{(2\pi\hbar)^d} \\ &= 2\omega^2(k)\frac{2v(k)}{\hbar^2 k}\int_0^\infty \int_0^{2\pi} \delta'\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|r\cos\theta\right) S_0\left(\frac{r}{\hbar}\right) \\ &\quad\times r^3[r\cos\theta(Q\mathbf{f}(|\mathbf{p}|))_1 + r\sin\theta(Q\mathbf{f}(|\mathbf{p}|))_2]\frac{d\theta dr}{(2\pi\hbar)^2} \\ &= 2\omega^2(k)\frac{2v(k)}{\hbar^2 k}\int_0^\infty \int_0^{2\pi} \delta'\left(\frac{2v(k)}{\hbar^2 k}|\mathbf{p}|r\cos\theta\right) \\ &\quad\times S_0\left(\frac{r}{\hbar}\right) r^4\cos\theta\frac{d\theta dr}{(2\pi\hbar)^2}(Q\mathbf{f}(|\mathbf{p}|))_1 \\ &= -4\omega^2(k)\left(\frac{2v(k)}{\hbar^2 k}\right)^{-1}(|\mathbf{p}|r)^{-2} \\ &\quad\times \int_0^\infty S_0\left(\frac{r}{\hbar}\right) r^4\frac{dr}{(2\pi\hbar)^2}(Q\mathbf{f}(|\mathbf{p}|))_1 \\ &= -\frac{m\hbar^2\omega^2(k)}{\pi^2|\mathbf{p}|^3}\int_0^\infty S_0(k_r)k_r^2 dk_r \mathbf{p}\cdot\mathbf{f}(|\mathbf{p}|). \end{aligned}$$

This completes the proof. ■

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