Adiabatic elimination for systems with inertia driven by compound Poisson colored noise

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We consider the dynamics of systems driven by compound Poisson colored noise in the presence of inertia. We study the limit when the frictional relaxation time and the noise autocorrelation time both tend to zero. We show that the Itô and Marcus stochastic calculuses naturally arise depending on these two time scales, and an extra intermediate type occurs when the two time scales are comparable. This leads to three different limiting regimes which are supported by numerical simulations. Furthermore, we establish that when the resulting compound Poisson process tends to the Wiener process in the frequent jump limit the Itô and Marcus calculuses, respectively, tend to the classical Itô and Stratonovich calculuses for Gaussian white noise, and the crossover type calculus tends to a crossover between the Itô and Stratonovich calculuses. Our results would be very helpful for understanding relevant experiments when jump type noise is involved.

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I. INTRODUCTION

The mathematical description of many physical problems contains variables obeying dynamics characterized by widely different time scales. One often pays more attention to the slow variables and thus the fast ones are eliminated in practice. This procedure is usually called adiabatic elimination and much work has been done in this field [1–4]. A typical example is the derivation of the Gaussian white noise limit for systems with inertia and multiplicative colored noise [3].

In various branches in natural and social sciences, stochastic processes driven by multiplicative non-Gaussian noise are common. This includes many examples such as the analysis of the shot noise in electrical circuits [5], the stock price modeling in option pricing [6], the stochastic modeling of soil salinity [7], and the small thermodynamic systems [8]. Dynamics subject to multiplicative noise have been studied over the last few decades [9,10]. The adiabatic elimination has been well developed for the case of Gaussian noise, but so far there are very few investigations for the non-Gaussian noise in a similar situation, to the best of the author's knowledge.

The instantaneous impulse in a driving process is often modeled as a white shot noise, and correspondingly an instantaneous change will be induced for the considered system. However, from the practical point of view, we want to emphasize that the impulse actually occurs in a very short but not an infinitely small time scale and the induced instantaneous change is indeed continuous instead of a pure jump. Though small, the time scale of the driving noise is finite and can be characterized through its autocorrelation time. Thus in the instantaneous impulse model we are simply dealing with an idealized limit in the mathematical modeling. This idea, which is common in physics and mathematics, has already been embodied in the famous Wong-Zakai smoothing limit for the Gaussian colored noise [11].

The presence of inertia introduces another characteristic time scale in the system, that is, the frictional relaxation time. In nondimensionalized form, the dynamics we will study in this paper reads

$$\epsilon^a \ddot{x} = -\dot{x} + b(x)\xi_\nu(t), \tag{1}$$

where $\xi_\nu(t)$ is assumed as the colored version of a white shot noise $\xi(t)$ with time scale $\nu \sim \mathcal{O}(\epsilon)$. Here a white shot noise is defined as the time derivative of a compound Poisson process $L(t)$ [12]. We will investigate the limiting equation describing the dynamics in coordinate space when both frictional relaxation time and noise autocorrelation time tend to zero.

The main results in this paper can be briefly stated as follows: depending on the magnitudes of the frictional relaxation time and the noise autocorrelation time, *three different regimes* will arise in the limiting dynamics.

A. Case 1

When the frictional relaxation time is smaller than the noise autocorrelation time ($a > 1$), the multiplicative noise in the limiting stochastic differential equation (SDE) should be interpreted in the sense of Marcus stochastic calculus:

$$dX(t) = b(X) \diamond dL(t). \tag{2}$$

B. Case 2

When the frictional relaxation time is larger than the noise autocorrelation time ($a < 1$), the limiting SDE should be interpreted in the sense of Itô stochastic calculus:

$$dX(t) = b(X) \cdot dL(t). \tag{3}$$

C. Case 3

When the frictional relaxation time and the noise autocorrelation time are comparable ($a = 1$), we obtain an intermediate type stochastic calculus denoted by $\star$:

$$dX(t) = b(X) \star dL(t). \tag{4}$$

The detailed definition for the above three kinds of SDEs will be stated in the next section.

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The important message here is that the limiting system will have different physical behaviors such as invariant distribution, stability thresholds, and so on depending on the two time scales. We should remark that the first two regimes have been found in [4] under special circumstances, but the intermediate regime (case 3) has never been demonstrated before. We also emphasize that the method we adopted here is quite different from [4] and is universal to some extent. We further consider the case when the resulting compound Poisson process tends to the Wiener process in the frequent jump limit. The three different obtained regimes then tend to the similar three limits in the Gaussian noise case derived in [3], which in turn confirms the validity of our result.

The rest of this paper is organized as follows. In Sec. II we present the model and make the adiabatic elimination in different parameter regimes. We also show some numerical examples to validate our analysis. In Sec. III we derive the limiting equations in the Gaussian white noise limit. Finally we make the conclusion.

II. POISSON COLORED NOISE LIMIT

A. Model setup

Consider the following Langevin equation with multiplicative white shot noise:

$$\tau \ddot{x} = -\dot{x} + b(x)\xi(t). \tag{5}$$

The parameter τ is the nondimensional relaxation time of the particle velocity. $b(x)$ is a sufficiently smooth function which is bounded together with its first two derivatives. $\xi(t)$ is a white shot noise with realizations:

$$\xi(t) = \sum_{i=1}^{N(t)} R_i \delta(t-\sigma_i). \tag{6}$$

Here σ_i is the random jump time with rate λ, R_i is the random jump size with distribution $p(r)$, $N(t)$ is the count of jumps until time t, and $\delta(t)$ is the Dirac δ function. We denote $L(t)$ the underlying compound Poisson process corresponding to $\xi(t)$, i.e., $L(t) = \sum_{i=1}^{N(t)} R_i H(t-\sigma_i)$, where $H(t)$ is the standard Heaviside function. We assume the random jump size R has distribution $p(r)$ and satisfies $\langle R \rangle = 0$ where the bracket denotes the ensemble average. The key point in understanding Eq. (5) is the definition of the stochastic integral $\int b(x)\xi(t)dt$. In the case of $\xi(t)$ being a Gaussian white noise, the well-known Itô and Stratonovich integrals are widely used in different disciplines. The Stratonovich integral is preferred by physicists due to the facts that it can be understood as the Wong-Zakai type smoothing limit [11] and that it obeys the Newton-Leibniz chain rule. This leads naturally to the idea that the instantaneous jumps are idealizations to smooth excitations during a very short time. This inspires us to consider a colored version of the white shot noise and consider the limit as the smoothing parameter goes to zero.

At first we define the smoothed Heaviside function $\theta(t)$ as

$$\theta(t) = \begin{cases} 0, & t < 0 \\ t, & t \in [0,1]. \\ 1, & t > 1 \end{cases} \tag{7}$$

Take $\xi_\nu(t) = \sum_{i=1}^{N(t)} R_i \dot{\theta}((t-\sigma_i)/\nu)$ as the colored version of $\xi(t)$ with parameter ν. Here the dot over θ means the derivative with respect to t. It is not difficult to see $\xi_\nu(t) \to \xi(t)$ in the pathwise sense as $\nu \to 0$, and the autocorrelation function of $\xi_\nu(t)$ is

$$\langle \xi_\nu(t)\xi_\nu(s)\rangle = \lambda\langle R^2\rangle \max\left\{\frac{1}{\nu^2}(\nu - |t-s|),0\right\}. \tag{8}$$

The derivations may be referenced in the Appendix. The autocorrelation function is nonzero only when $|t-s| < \nu$, and thus the parameter ν represents the characteristic autocorrelation time of the smoothed white shot noise. In the following contexts we assume the autocorrelation time is much smaller than the time between two jumps. We are interested in studying the limit of Eq. (5) when τ as well as ν tend to zero. To this end, we introduce the parameter ϵ as $\tau = \tau_0 \epsilon^a$ and $\nu = \nu_0 \epsilon^c$, where $a,c > 0$, τ_0, and ν_0 are of order 1, and $\epsilon \ll 1$.

B. Single jump case

Since we assume that the autocorrelation time of the colored noise is much smaller than the time between two jumps, we can focus on each single jump separately. We consider a special realization of the Poisson colored noise with only one jump at time $t=0$ and assume the jump size is R. We have

$$\xi_\nu(t) = \begin{cases} R/\nu, & 0 \leqslant t \leqslant \nu \\ 0, & t > \nu \end{cases}. \tag{9}$$

Without loss of generality we take $\tau_0 = 1$ and $c = 1$ in the analysis. The other cases just correspond to a rescaling of parameters and it will not affect the results. Then the equation becomes

$$\epsilon^a \ddot{x} = -\dot{x} + b(x)\xi_\nu(t). \tag{10}$$

Define $y(t) = \dot{x}(t)$, $x_0 = x(0)$, and $y_0 = y(0)$. By taking advantage of the variation of constant, we get

$$\dot{x}(t) = y_0 \exp\left(-\frac{t}{\epsilon^a}\right) + \frac{1}{\epsilon^a}\int_0^t b[x(s)]\xi_\nu(s)\exp\left(-\frac{t-s}{\epsilon^a}\right)ds. \tag{11}$$

After integration by parts, we obtain the equation for the coordinates:

$$\begin{aligned} x(t) &= x_0 + y_0\epsilon^a\left[1-\exp\left(-\frac{t}{\epsilon^a}\right)\right] \\ &\quad + \int_0^t b[x(s)]\xi_\nu(s)\left[1-\exp\left(-\frac{t-s}{\epsilon^a}\right)\right]ds \\ &= x_0 + y_0\epsilon^a\left[1-\exp\left(-\frac{t}{\epsilon^a}\right)\right] \\ &\quad + \int_0^{t\wedge\nu_0\epsilon} b[x(s)]\frac{R}{\nu_0\epsilon}\left[1-\exp\left(-\frac{t-s}{\epsilon^a}\right)\right]ds, \end{aligned} \tag{12}$$

where we get the second equality by substituting Eq. (9) into the equation and $t \wedge s = \min\{t,s\}$. Clearly we have $y_0\epsilon^a[1-\exp(-t/\epsilon^a)] = \mathcal{O}(\epsilon^a) \to 0$ as $\epsilon \to 0$. Since we assume the autocorrelation time of the colored noise is much smaller than the time between two jumps, we mainly focus on the coordinate $x(t)$ on a larger time scale $[0,T]$ and at the same time T itself is small enough, which means

$$T \gg \max\{\epsilon^a,\epsilon\} \quad \text{and} \quad T \to 0$$

as $\epsilon \to 0+$. Now let us introduce the function $z_\epsilon(t)$ which satisfies

$$z_\epsilon(t) = x_0 + R\int_0^t b[z_\epsilon(s)]\left[1 - \exp\left(-\frac{\nu_0(t-s)}{\epsilon^{a-1}}\right)\right]ds \tag{13}$$

when $0 \leqslant t \leqslant 1$. It is not difficult to find that $z_\epsilon(t) = x(\nu_0\epsilon t) + o(1)$ for $t \in [0,1]$ since they only differ from an $\mathcal{O}(\epsilon^a)$ term in Eq. (12). The assumption on T leads to

$$x(T) = x_0 + R\int_0^1 b[z_\epsilon(s)]ds + o(1). \tag{14}$$

We obtain the following three types of limits by analyzing the behavior of $z_\epsilon(t)$.

1. Case 1: Marcus type limiting equation

If $a > 1$, $\exp(-1/\epsilon^{a-1})$ tends to zero exponentially as $\epsilon \to 0$ and thus $z_\epsilon(t) = z(t) + o(1)$, where $z(t)$ satisfies

$$z(t) = x_0 + R\int_0^t b[z(s)]ds. \tag{15}$$

Correspondingly from Eq. (14) we obtain

$$x(T) = z(1) + o(1) \tag{16}$$

as $\epsilon \to 0$. With another notation, we have the following ordinary differential equation (ODE) for z:

$$\begin{aligned} &\dot{z}(t) = Rb[z(t)], \quad 0 \leqslant t \leqslant 1 \\ &z(0) = x_0. \end{aligned} \tag{17}$$

We denote the final reduced limiting equation for x as

$$dX(t) = b(X) \diamond dL(t), \quad X(0) = x_0. \tag{18}$$

The limiting variable X will experience an instantaneous jump at the jump time of $\xi(t)$. From Eq. (14), the jump size of X is given by

$$\Delta X = R\int_0^1 b[z(t)]dt, \tag{19}$$

where $z(t)$ satisfies Eq. (17).

The ODE Eq. (17) is called Marcus mapping and the corresponding stochastic calculus Eq. (18) is called Marcus canonical calculus (see [13] for more details), which was pioneered by Marcus [14,15]. Recently, it has been discussed in the field of small thermodynamic system in [8,16], and further developments on its connection with the Wong-Zakai smoothing limit and the numerical simulations are studied in [17]. The Marcus canonical integral can be understood as an extension of the Stratonovich integral to non-Gaussian processes.

In this parameter regime, the frictional relaxation time is small compared to the noise autocorrelation time, so the noise can be indeed viewed as a smoothed process and it naturally results in the Marcus integral which is the Wong-Zakai type smoothing limit for non-Gaussian processes.

2. Case 2: Itô type limiting equation

If $a < 1$, ϵ^{1-a} tends to zero as $\epsilon \to 0$ and thus Eq. (13) leads to

$$z_\epsilon(t) = z(t) + \mathcal{O}(\epsilon^{1-a}) \tag{20}$$

and $z(t) \equiv x_0$. Correspondingly we obtain

$$x(T) = x_0 + b(x_0)R + \mathcal{O}(\epsilon^{\min(a,1-a)}). \tag{21}$$

Similar to the previous case, we can define

$$\begin{aligned} &\dot{z}(t) = 0, \quad 0 \leqslant t \leqslant 1 \\ &z(0) = x_0 \end{aligned} . \tag{22}$$

The final reduced limiting equation for x is denoted as

$$dX(t) = b(X) \cdot dL(t), \quad X(0) = x_0, \tag{23}$$

where X will experience an instantaneous jump at the jump time of $\xi(t)$, and the jump size is simply

$$\Delta X = R\int_0^1 b[z(t)]dt = b(x_0)R. \tag{24}$$

The definition Eq. (23) is exactly the classical Itô type calculus for white shot noise. This can be intuitively explained, as when the frictional relaxation time is much larger than the noise autocorrelation time the smoothing through relaxation has no chance to take effect on the overall dynamics. Thus we get the Itô type limit.

3. Case 3: Crossover type limiting equation

If $a = 1$, Eq. (13) becomes ϵ independent and thus

$$z(t) = x_0 + R\int_0^t b[z(s)]\{1 - \exp[-\nu_0(t-s)]\}ds. \tag{25}$$

Correspondingly we can define the following crossover type ODE mapping:

$$\begin{aligned} &\ddot{z}(t) = -\nu_0\dot{z}(t) + R\nu_0 b[z(t)], \quad 0 \leqslant t \leqslant 1 \\ &z(0) = x_0, \quad \dot{z}(0) = 0. \end{aligned} \tag{26}$$

We denote the final reduced limiting equation for x as

$$dX(t) = b(X) \star dL(t), \quad X(0) = x_0, \tag{27}$$

where X will experience an instantaneous jump at the jump time of $\xi(t)$. The jump size is also

$$\Delta X = R\int_0^1 b[z(t)]dt, \tag{28}$$

where $z(t)$ satisfies Eq. (26).

In summary, when the frictional relaxation time is comparable to the noise autocorrelation time, a new mechanism arises. The reduced dynamics cannot be understood as either Itô calculus or Marcus calculus. The crossover type dynamics depends on the constant ν_0, which is quite similar to the case in [3].

We also remark here that we can formally derive the limiting equations Eqs. (17) and (22) from Eq. (26) by varying ν_0. We utilize the singular perturbation analysis [18,19] to do the job. Taking the limit $\nu_0 \to 0$ which corresponds to the regime $a < 1$, we expand the solution in a power series:

$$z(t) = z_0(t) + \nu_0 z_1(t) + \nu_0^2 z_2(t) + \cdots . \tag{29}$$

Substituting this expansion into Eq. (26) we obtain the leading order term $\ddot{z}_0(t) = 0$. Together with its initial condition $\dot{z}_0(0) = 0$, $z_0(0) = x_0$, we have $z_0(t) = x_0$. This exactly gives Eq. (22). Taking the limit $\nu_0 \to \infty$ which corresponds to the regime $a > 1$, we similarly expand

$$z(t) = z_0(t) + \frac{1}{\nu_0} z_1(t) + \frac{1}{\nu_0^2} z_2(t) + \cdots \tag{30}$$

and substitute this into Eq. (26). The leading order equation is $\dot{z}_0(t) = Rb[z_0(t)]$, which is exactly the ODE mapping Eq. (17).

Our analysis is supported by numerical simulations. For simplicity, we fix the jump time at $t = 1$ and 2, and the jump sizes are both 2. Some other parameters are $\nu_0 = 1$, $\epsilon = 0.02, b(x) = x$, and $x(0) = 1$. The second order Runge-Kutta method is adopted in simulating both the jump ODEs and the Langevin equation with smoothed noise. We choose a different parameter a to test our analytical result. Figures 1(a)–1(c) correspond to the limiting dynamics interpreted with Itô type, crossover type, and Marcus type calculuses, respectively. All the numerical results clearly confirm the validity of our theoretical analysis.

C. Generalization

The systematic adiabatic elimination procedure described in the previous section can be extended in a straightforward way to cover the multidimensional case in $\mathbb{R}^d$ with inertia and colored noise. Consider the following high dimensional Langevin equation:

$$\epsilon^a \ddot{\boldsymbol{x}} = -\dot{\boldsymbol{x}} + \mathbf{b}(\boldsymbol{x})\boldsymbol{\xi}(t), \tag{31}$$

where $\boldsymbol{x} \in \mathbb{R}^d, \mathbf{b}(\boldsymbol{x}) \in \mathbb{R}^{d\times n}$, and $\boldsymbol{\xi}(t) = \sum_{i=1}^{N(t)} \boldsymbol{R}_i \delta(t - \sigma_i)$ with $\boldsymbol{R}_i \in \mathbb{R}^n$ and we again write $\boldsymbol{L}(t)$ for the underlying compound Poisson process of $\boldsymbol{\xi}(t)$. Similarly we consider the colored noise $\boldsymbol{\xi}_\nu(t) = \sum_{i=1}^{N(t)} \boldsymbol{R}_i \dot{\theta}[(t - \sigma_i)/\nu]$. Using a similar approach as in the previous section, one can prove the following results with a unified form.

1. *Case 1: Marcus type limiting equation*

When $a > 1$, the solution of the limiting equation is

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \sum_{i=1}^{N(t)} [\Phi^M(\boldsymbol{X}(\sigma_i-), \boldsymbol{R}_i) - \boldsymbol{X}(\sigma_i-)], \tag{32}$$

where $\Phi^M(\boldsymbol{X}_0, \boldsymbol{R}) : z(0) \to z(1)$ is the Marcus mapping defined as

$$\begin{aligned} \dot{z}(t) &= \mathbf{b}[z(t)]\boldsymbol{R}, \quad 0 \leqslant t \leqslant 1 \\ z(0) &= \boldsymbol{X}_0. \end{aligned} \tag{33}$$

We take the notation $\diamond$ for the above Marcus calculus as

$$d\boldsymbol{X}(t) = \mathbf{b}[\boldsymbol{X}(t)] \diamond d\boldsymbol{L}(t). \tag{34}$$

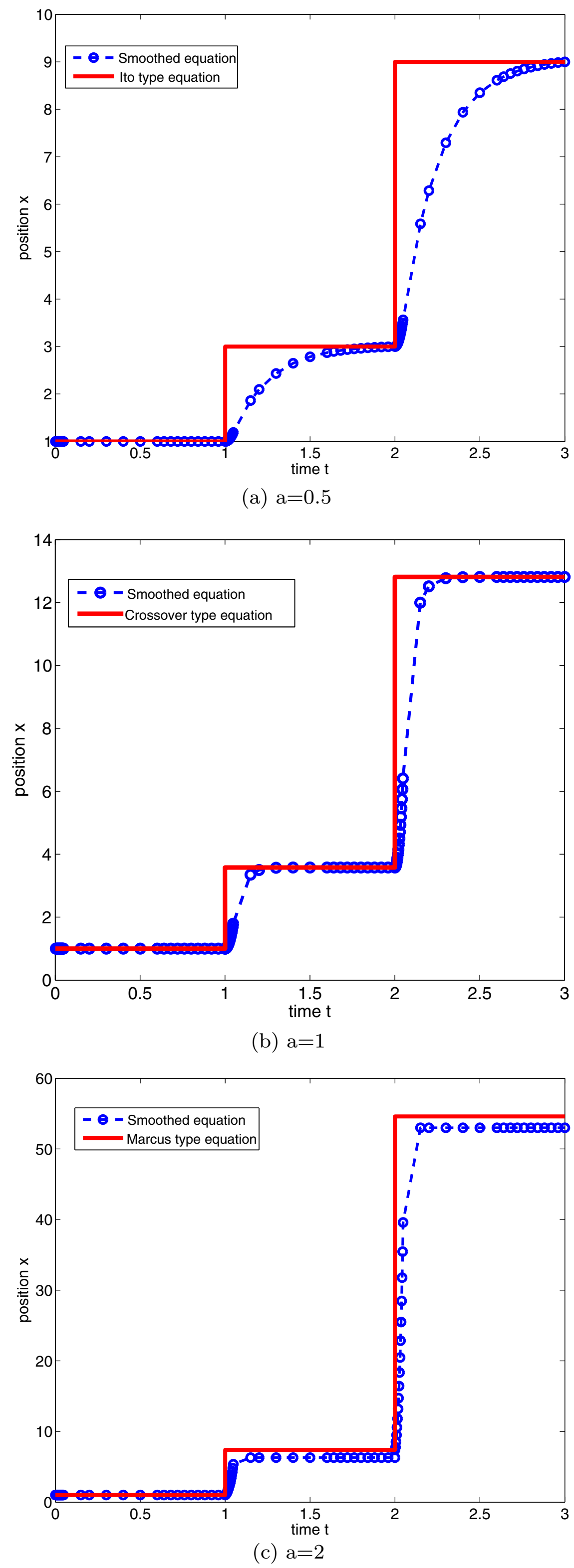

FIG. 1. (Color online) Numerical comparison between the reduced dynamics and the Langevin dynamics. The jump time is fixed to be $t = 1$ and 2, and both jump sizes equal 2. We choose $\epsilon = 0.02$ and initial position $x(0) = 1$. Different choices of a are taken to confirm our analysis. The second order Runge-Kutta method is adopted for the numerical simulations. The solid red line and dashed blue line with circles correspond to the numerical solution for the reduced dynamics and Langevin equation with colored noise, respectively.

2. Case 2: Itô type limiting equation

When $a < 1$, the solution of the limiting equation is

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \sum_{i=1}^{N(t)} [\Phi^I(\boldsymbol{X}(\sigma_i-),\boldsymbol{R}_i) - \boldsymbol{X}(\sigma_i-)], \quad (35)$$

where $\Phi^I(\boldsymbol{X}_0,\boldsymbol{R}) : z(0) \to z(1)$ is the Itô mapping defined as

$$\begin{aligned} \dot{z}(t) &= \mathbf{b}(\boldsymbol{X}_0)\boldsymbol{R}, \quad 0 \leqslant t \leqslant 1 \\ z(0) &= \boldsymbol{X}_0. \end{aligned} \quad (36)$$

We take the notation $\cdot$ for the Itô calculus as

$$d\boldsymbol{X}(t) = \mathbf{b}[\boldsymbol{X}(t)] \cdot d\boldsymbol{L}(t). \quad (37)$$

3. Case 3: Crossover type limiting equation

When $a = 1$, the solution of the limiting equation is

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \sum_{i=1}^{N(t)} [\Phi^C(\boldsymbol{X}(\sigma_i-),\mathbb{R}_i) - \boldsymbol{X}(\sigma_i-)], \quad (38)$$

where the mapping $\Phi^C(\boldsymbol{X}_0,\boldsymbol{R}) : z(0) \to z(1)$ is defined as the solution mapping of

$$\begin{aligned} \dot{z}(t) &= \mathbf{b}[\boldsymbol{y}(t)], \quad 0 \leqslant t \leqslant 1 \\ \ddot{\boldsymbol{y}}(t) &= -\nu_0\dot{\boldsymbol{y}}(t) + \nu_0\mathbf{b}[\boldsymbol{y}(t)]\boldsymbol{R}, \quad 0 \leqslant t \leqslant 1 \\ z(0) &= \boldsymbol{X}_0 \\ \boldsymbol{y}(0) &= \boldsymbol{X}_0 \\ \dot{\boldsymbol{y}}(0) &= 0. \end{aligned} \quad (39)$$

We take the notation $\star$ for the crossover type calculus as

$$d\boldsymbol{X}(t) = \mathbf{b}[\boldsymbol{X}(t)] \star d\boldsymbol{L}(t). \quad (40)$$

III. FROM WHITE SHOT NOISE TO GAUSSIAN WHITE NOISE

The compound Poisson process will converge to the Wiener process if the jump rate λ tends to infinity while keeping $\langle R\rangle = 0,\lambda\langle R^2\rangle = 1$. We introduce a small parameter $\kappa \to 0$ and let $\lambda = \kappa^{-2}, R = \kappa\hat{R}$ where $\hat{R}$ is a random variable that satisfies $\langle\hat{R}\rangle = 0,\langle\hat{R}^2\rangle = 1$. Now we investigate how the behavior of our results in Sec. II will change in this frequent jump limit. It is known that the limiting equation for Gaussian white noise has been fully discussed in [3]. We will show there is a correspondence for the three regimes between the white shot noise case and the Gaussian white noise case in this section. For simplicity we will only derive our results for the one dimensional case here but all of the results can be easily generalized to the high dimensional case.

Consider a general jump mapping $\Phi(\cdot,R) : \mathbb{R} \to \mathbb{R}$ and the stochastic calculus for $\dot{X}(t) = b(X(t))\xi(t)$ defined as

$$X(t) = X(0) + \sum_{i=1}^{N(t)} [\Phi(X(\sigma_i-),R_i) - X(\sigma_i-)]. \quad (41)$$

We can get the infinitesimal generator for X as

$$\begin{aligned} \mathcal{L}f(x) &= \lim_{t\to 0} \frac{\langle f(X(t))\rangle_x - f(x)}{t} \\ &= \lim_{t\to 0}\left\langle \frac{[(f(\Phi(x,R)) - f(x))\lambda t + \mathcal{O}(t^2)]\exp(-\lambda t)}{t}\right\rangle \\ &= \lambda\langle f(\Phi(x,R)) - f(x)\rangle, \end{aligned} \quad (42)$$

where the notation $\langle\cdot\rangle_x$ is the ensemble average with respect to initial state x and the expectation $\langle\cdot\rangle$ in the last two equalities is with respect to the jump variable R.

A. Marcus type

We use the singular perturbation analysis to get the frequent jump limit [18,19]. For the Marcus integral, the solution to Eq. (33) can be expanded as $z = z_0 + \kappa\hat{R}z_1 + \kappa^2\hat{R}^2 z_2 + \cdots$ when κ is small. Substituting this into the equation gives a hierarchy of equations:

$$\begin{aligned} &\dot{z}_0 = 0, && z_0(0) = x, \\ &\dot{z}_1 = b(z_0), && z_1(0) = 0, \\ &\dot{z}_2 = \dot{b}(z_0)z_1, && z_2(0) = 0, \\ &\cdots && \cdots. \end{aligned} \quad (43)$$

This gives

$$\Phi^M(x,R) = x + \kappa\hat{R}b(x) + \frac{\kappa^2\hat{R}^2}{2}b(x)\dot{b}(x) + \cdots. \quad (44)$$

Substituting this into Eq. (42) we obtain

$$\begin{aligned} \mathcal{L}^M f(x) &= \lambda\Big\langle \dot{f}(x)\Big(b(x)\kappa\hat{R} + \frac{\kappa^2\hat{R}^2}{2}b(x)\dot{b}(x) + \cdots\Big) \\ &\quad + \frac{1}{2}\ddot{f}(x)\Big(b(x)\kappa\hat{R} + \cdots\Big)^2 + \cdots\Big\rangle \\ &\to \frac{1}{2}\Big(b(x)\frac{\partial}{\partial x}\Big)^2 f(x) \quad \text{as} \quad \kappa \to 0. \end{aligned} \quad (45)$$

Thus we have the Fokker-Planck equation:

$$\frac{\partial P^M(x,t)}{\partial t} = (\mathcal{L}^M)^* P^M(x,t) = \frac{1}{2}\Big(\frac{\partial}{\partial x}b(x)\Big)^2 P^M(x,t), \quad (46)$$

where $(\mathcal{L}^M)^*$ stands for the adjoint operator of $\mathcal{L}^M$. This Fokker-Planck equation corresponds to the SDE:

$$dX(t) = b(X(t)) \circ dW(t), \quad (47)$$

where $W(t)$ is a Wiener process and the notation $\circ$ represents the Stratonovich integral. This corresponds to the Stratonovich regime in [3].

B. Itô type

For the Itô type integral, the solution to Eq. (36) can be expanded as $z = z_0 + \kappa\hat{R}z_1 + \kappa^2\hat{R}^2 z_2 + \cdots$. Substituting this into the equation gives a hierarchy of equations:

$$\begin{aligned} &\dot{z}_0 = 0, && z_0(0) = x, \\ &\dot{z}_1 = b(z_0), && z_1(0) = 0, \\ &\dot{z}_i = 0, && z_i(0) = 0, \quad i \geqslant 2. \end{aligned} \quad (48)$$

This gives $\Phi^I(x,R) = x + \kappa \hat{R} b(x)$. Substituting this into Eq. (42) we obtain

$$\mathcal{L}^I f(x) = \lambda\left\langle \dot{f}(x)b(x)\kappa\hat{R} + \frac{\kappa^2\hat{R}^2}{2}\ddot{f}(x)b^2(x) + \mathcal{O}(\kappa^3)\right\rangle \to \frac{1}{2}\ddot{f}(x)b^2(x) \quad \text{as} \quad \kappa \to 0. \tag{49}$$

Thus we have the Fokker-Planck equation:

$$\frac{\partial P^I(x,t)}{\partial t} = (\mathcal{L}^I)^* P^I(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(b^2(x)P^I(x,t)), \tag{50}$$

and the corresponding SDE is

$$dX(t) = b[X(t)] \cdot dW(t), \tag{51}$$

where $W(t)$ is a Wiener process and the notation $\cdot$ represents the Itô integral. This corresponds to the Itô regime in [3].

C. Crossover type

For the intermediate type calculus, we similarly expand the solution to Eq. (39) as $z = z_0 + \kappa\hat{R}z_1 + \kappa^2\hat{R}^2 z_2 + \cdots$. Substituting this into the equation gives a hierarchy of equations:

$$\begin{aligned} &\ddot{z}_0 = -\nu_0\dot{z}_0, && z_0(0) = x, \quad \dot{z}_0(0) = 0,\\ &\ddot{z}_1 = -\nu_0\dot{z}_1 + \nu_0 b(z_0), && z_1(0) = 0, \quad \dot{z}_1(0) = 0,\\ &\cdots && \cdots. \end{aligned} \tag{52}$$

This gives

$$z_0(t) = x, \quad z_1(t) = \frac{e^{-\nu_0 t} - 1 + \nu_0 t}{\nu_0} b(x). \tag{53}$$

We have

$$\begin{aligned} \Phi^C(x,R) &= x + \kappa\hat{R}\int_0^1 b(z)ds\\ &= x + \kappa\hat{R}\int_0^1 [b(x) + \dot{b}(x)(\kappa\hat{R}z_1 + \cdots) + \cdots]ds\\ &= x + \kappa\hat{R}b(x) + \kappa^2\hat{R}^2\dot{b}(x)\int_0^1 z_1(t)dt\\ &= x + \kappa\hat{R}b(x) + \alpha\kappa^2\hat{R}^2 b(x)\dot{b}(x) + \cdots, \end{aligned} \tag{54}$$

where

$$\alpha = \frac{1 - e^{-\nu_0} - \nu_0 + \nu_0^2/2}{\nu_0^2} \in \left(0, \frac{1}{2}\right). \tag{55}$$

Substituting this into Eq. (42) we obtain the infinitesimal generator for our crossover $\star$ calculus:

$$\begin{aligned} &\mathcal{L}^C f(x)\\ &= \lambda\left\langle \dot{f}(x)(b(x)\kappa\hat{R} + \alpha\kappa^2\hat{R}^2 b(x)\dot{b}(x) + \cdots)\right.\\ &\quad \left. + \frac{1}{2}\ddot{f}(x)(b(x)\kappa\hat{R} + \alpha\kappa^2\hat{R}^2 b(x)\dot{b}(x) + \cdots)^2 + \cdots\right\rangle\\ &\to \frac{1}{2}\ddot{f}(x)b^2(x) + \alpha\dot{f}(x)b(x)\dot{b}(x)\\ &= (1-2\alpha)\mathcal{L}^I f(x) + 2\alpha\mathcal{L}^M f(x) \quad \text{as} \quad \kappa \to 0. \end{aligned} \tag{56}$$

Thus we have the Fokker-Planck equation:

$$\begin{aligned} \frac{\partial P^C(x,t)}{\partial t} &= (\mathcal{L}^C)^* P^C(x,t)\\ &= (1-2\alpha)(\mathcal{L}^I)^* P^C(x,t) + 2\alpha(\mathcal{L}^M)^* P^C(x,t), \end{aligned} \tag{57}$$

which corresponds to the SDE:

$$dX(t) = (1-2\alpha)b[X(t)] \cdot dW(t) + 2\alpha b[X(t)] \circ dW(t), \tag{58}$$

where $W(t)$ is a Wiener process and the notations $\cdot$ and $\circ$ represent the Itô and Stratonovich integrals, respectively. This result corresponds to the crossover regime in [3]. The final limiting equation is a combination of Itô and Stratonovich integrals, and their weight depends on the parameter α which is related to ν_0 (the ratio between the relaxation time and noise autocorrelation time).

Additionally, as we have done in Sec. II B, we can formally derive the Itô type Eq. (47) and Marcus type Eq. (51) from the crossover type Eq. (58) by varying parameter ν_0. Taking $\nu_0 \to \infty$, which corresponds to the regime $a > 1$, we have $\alpha \to 1/2$ and the crossover type calculus Eq. (58) reduces to the Stratonovich calculus Eq. (47). Taking $\nu_0 \to 0$, which corresponds to the regime $a < 1$, we have $\alpha \to 0$ and the crossover type calculus Eq. (58) reduces to the Itô calculus Eq. (51).

IV. CONCLUSION

We have shown the results of adiabatic elimination for the Langevin equation with inertia driven by multiplicative white shot noise. The final limiting equation depends on the magnitude of frictional relaxation time and noise autocorrelation time. The multiplicative noise in the limiting equation can be described by either Itô or Marcus type calculus depending on whether the relaxation time is larger or smaller than the noise autocorrelation time. Furthermore, a new type of stochastic integral is found when the two time scales are comparable. Our results are consistent with the earlier work when we take the Gaussian white noise limit for the white shot noise [3]. Furthermore our approach is based on singular perturbation analysis which is general.

What we want to emphasize here is that either the Itô or Marcus integral could be the right integral in realistic problems. The Itô integral is nonanticipating and thus preserves causality [12], while the Marcus integral obeys the Newton-Leibniz chain rule (like the Stratonovich integral) and could be understood as the result of the Wong-Zakai type smoothing limit [17]. Whether the Itô integral or Marcus integral should be preferred depends on the realistic problems to be treated. This is quite similar to the choice between the Itô and the Stratonovich integral in other contexts. Nonanticipation is also an important property in financial mathematics [6], while the Newton-Leibniz chain rule is important in constructing thermodynamics laws in stochastic energetics [8,17]. Researchers should choose the right integral based on the realistic problems and their concrete setup. Our paper offers a view that the parameter regime could affect the limiting stochastic integral, which is a common hidden mechanism for many physical problems.

It will be instructive to shortly discuss the Marcus integral and the other definitions of the Stratonovich integral for the jump type noise here. To our knowledge, the Stratonovich integral is first defined for Gaussian white noise [20]. It obeys the Newton-Leibniz chain rule and can be viewed as the Wong-Zakai smoothing limit which brings natural physical interpretation to it. These good properties inspire scholars to extend the Stratonovich integral to more general processes and much work has been done [6,12,13]. Interestingly, there is no unified definition for the Stratonovich integral extended to the more general noise. It has been observed in [13] (line 3, p. 238) that the straightforward extension by taking the weight 1/2 for both the left-most and right-most endpoints in the discretized integral does not lead to the Newton-Leibniz chain rule for the Lévy type noise. Another choice, taking the midpoint for the integrand, does not give the Newton-Leibniz rule either [8], while the Marcus integral considered here is a good candidate to achieve this goal. This property has been utilized in [8,17] to understand the stochastic energetics for small systems. Furthermore, the current paper gives the rationale of under what circumstances which kind of stochastic integral is preferred from the adiabatic elimination point of view. This is different from the usual Wong-Zakai type smoothing limit argument [17]. Interested readers are referred to [8,13,17,21,22] for more details.

Finally we want to remark that the three different limiting equations discussed have different properties depending on the interpretation of the multiplicative noise. This informs us that great care should be taken in the adiabatic elimination procedure when there are more than one fast time scale variables, as indicated in [3]. Detailed inspection is necessary to ensure our correctly capturing the real physical processes through reduced models.

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APPENDIX: AUTOCORRELATION FUNCTION

Here we prove the autocorrelation function of $\xi_\nu(t)$ is

$$\langle \xi_\nu(t)\xi_\nu(s)\rangle = \max\left\{\frac{\lambda\langle R^2\rangle}{\nu^2}(\nu - |t-s|),0\right\}. \tag{A1}$$

At first we assume $s < t$. Using the definition of $\theta(t)$, we have

$$\dot{\theta}(t) = \sum_{i=1}^{N(t)} \frac{R_i}{\nu} 1_{[\sigma_i,\sigma_i+\nu]}(t). \tag{A2}$$

Here $1_A(x)$ is the characteristic function defined as $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Substituting the above equation into $\langle \xi_\nu(t)\xi_\nu(s)\rangle$, we have

$$\langle \xi_\nu(t)\xi_\nu(s)\rangle = \left\langle \sum_{i=1}^{N(t)}\sum_{j=1}^{N(s)} \frac{R_i R_j}{\nu^2} 1_{[\sigma_i,\sigma_i+\nu]}(t) 1_{[\sigma_j,\sigma_j+\nu]}(s)\right\rangle. \tag{A3}$$

Recall that the random jump size R satisfies $\langle R\rangle = 0$, and R_i is independent of R_j. This leads to $\langle R_i R_j\rangle = \langle R^2\rangle\delta_{ij}$, where $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$. Together with the fact that the random jump time σ is independent of the random jump size R, we have

$$\begin{aligned}\langle \xi_\nu(t)\xi_\nu(s)\rangle &= \frac{\langle R^2\rangle}{\nu^2}\left\langle \sum_{i=1}^{N(t)} 1_{[\sigma_i,\sigma_i+\nu]}(t) 1_{[\sigma_j,\sigma_j+\nu]}(s)\right\rangle \\ &= \begin{cases} \frac{\langle R^2\rangle}{\nu^2}\left\langle \sum_{i=1}^{N(t)} 1_{[t-\nu,s]}(\sigma_i)\right\rangle, & |t-s| \leqslant \nu \\ 0, & |t-s| > \nu \end{cases}.\end{aligned} \tag{A4}$$

To calculate the average in the above equation, we need Theorem 5.2 from [23]. It is stated as below:

Lemma 1. Given that $N(t) = n$, the n arrival times $\sigma_1,\ldots,\sigma_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0,t)$.

Thus conditioned on given $N(t) = n$, we have

$$\begin{aligned}&\left\langle \sum_{i=1}^{N(t)} 1_{[t-\nu,s]}(\sigma_i)|N(t) = n\right\rangle \\ &= \int\cdots\int_{0\leqslant x_1\leqslant\cdots\leqslant x_n\leqslant t} \sum_{i=1}^{n} 1_{[t-\nu,s]}(x_i)\frac{n!}{t^n}dx_1\cdots dx_n \\ &= \frac{1}{n!}\int\cdots\int_{0\leqslant x_i\leqslant t, i=1,\ldots,n} \sum_{i=1}^{n} 1_{[t-\nu,s]}(x_i)\frac{n!}{t^n}dx_1\cdots dx_n \\ &= \frac{1}{t^n}\sum_{i=1}^{n}\int\cdots\int_{0\leqslant x_i\leqslant t, i=1,\ldots,n} 1_{[t-\nu,s]}(x_i)\frac{n!}{t^n}dx_1\cdots dx_n \\ &= \frac{1}{t^n} n t^{n-1}(s-t+\nu) \\ &= \frac{n}{t}(s-t+\nu).\end{aligned} \tag{A5}$$

Here the second equation is obtained by observing that x_i and x_j are exchangeable. Taking an expectation with respect to $N(t)$, we have

$$\left\langle \sum_{i=1}^{N(t)} 1_{[t-\nu,s]}(\sigma_i)\right\rangle = \frac{\lambda t}{t}(s-t+\nu) = \lambda(s-t+\nu). \tag{A6}$$

Substituting this into Eq. (A4) we get Eq. (A1).

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