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LARGE DEVIATIONS FOR TWO-SCALE CHEMICAL KINETIC PROCESSES*

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Abstract. We formulate the large deviations for a class of two scale chemical kinetic processes motivated from biological applications. The result is successfully applied to treat a genetic switching model with positive feedbacks. The corresponding Hamiltonian is convex with respect to the momentum variable as a byproduct of the large deviation theory. This property ensures its superiority in the rare event simulations compared to the result obtained by formal WKB asymptotics. The result is of general interest in understanding the large deviations for multiscale problems.

Key words. Chemical kinetic processes, two-scale large deviations, rare events, genetic switching.

AMS subject classifications. 60F10, 60J75, 62P10, 92C37.

1. Introduction

We will investigate the large deviations for a class of two-scale chemical kinetic processes with the slow variable $\boldsymbol{z}_n \in \mathbb{N}^d/n$ satisfying

$$\boldsymbol{z}_n(t) = \boldsymbol{z}_n(0) + \sum_{i=1}^{S} \frac{1}{n} P_i \left(n \int_0^t \lambda_i(\boldsymbol{z}_n(s), \xi_n(s)) ds \right) \boldsymbol{u}_i \tag{1.1}$$

subject to some fixed initial state $\boldsymbol{z}_n(0) = \boldsymbol{z}^0$, where $\{P_i(t)\}_{i=1,\dots,S}$ are independent uni-rate Poisson processes, $\lambda_i \in \mathbb{R}^+$ is called the propensity function which characterizes the reaction rate of the ith reaction, and $\boldsymbol{u}_i \in \mathbb{Z}^d$ is called the state change vector. The number $n \in \mathbb{N}$ corresponds to the system volume; thus, $\boldsymbol{z}_n$ has the meaning of concentration (number of molecules per volume) for the considered kinetic system. The fast variable $\xi_n \in \mathbb{Z}_D := \{1,2,\dots,D\}$ is a simple jump process with the time-dependent rate $nq_{ij}(\boldsymbol{z}_n(t))$ from state i to j at time t. With this mathematical setup, the processes $\boldsymbol{z}_n(t)$ and $\xi_n(t)$ are fully coupled to each other and the infinitesimal generator $\mathcal{L}_n$ of this system has the form

$$\mathcal{L}_n h(\boldsymbol{z},i) = n \sum_{l=1}^{S} \lambda_l(\boldsymbol{z},i)[h(\boldsymbol{z}+\boldsymbol{u}_l/n,i) - h(\boldsymbol{z},i)] + n \sum_{\substack{j=1 \\ j \neq i}}^{D} q_{ij}(z)[h(\boldsymbol{z},j) - h(\boldsymbol{z},i)], \tag{1.2}$$

where $\boldsymbol{z} \in \mathbb{N}^d/n$, $i \in \mathbb{Z}_D$, and h is any compactly supported smooth function of $\boldsymbol{z}$ for each i. For more about the notations and the backgrounds on the chemical kinetic processes, the readers may be referred to [11, 13].

The above problem is motivated by our recent rare event study in the biological applications [1, 19, 21]. In a cell, the reactions underlying gene expression usually involve a low copy number of molecules, such as DNA, mRNAs, and transcription factors,

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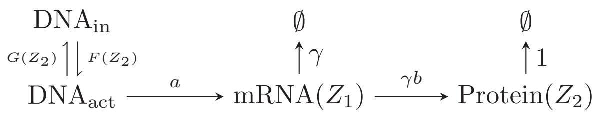
so the stochasticity in the gene regulation process is inevitable even under constant environmental conditions [10]. When the number of the molecules for all species goes to infinity and the law of mass action holds for the propensity functions, one gets the well-known large volume limit or Kurtz's limit, which gives the deterministic reaction rate equations for the concentration of the species [17]. The convergence result can be further refined to the large deviation type [26]. Recently, the following typical biological model with positive feedbacks is utilized to investigate the robustness of the genetic switching system [1, 4, 19, 21]. This problem is a special case of our formulation shown at the



beginning of this paper for $d=2$, $D=2$, and $S=4$. Denote by n the system size and $\boldsymbol{z}=(z_1,z_2)=(Z_1,Z_2)/n$ the slow variables after taking large volume scaling, where Z_1 and Z_2 are the number of mRNA and protein molecules, respectively. Since there is only one molecule of DNA in an active ($\mathrm{DNA_{act}}$) or an inactive state ($\mathrm{DNA_{in}}$), for better use of notation, we take the fast variable $\xi\in\{0,1\}$ instead of $\{1,2\}$ to represent that the DNA is in an inactive ($\xi=0$) or an active state ($\xi=1$), respectively. By taking into account the scaling of parameters

$$a\sim nb^{-1}, \quad F(Z_2),G(Z_2)\sim O(n) \quad \text{if} \quad Z_1,Z_2\sim O(n),$$

we further assume

$$F(Z_2)=nf(z_2), \quad G(Z_2)=ng(z_2). \tag{1.3}$$

This assumption holds when we consider Hill-function type jump rates with Hill coefficient 2 and large volume scaling for equilibrium constants [1]. Thus, we have the rescaled jump rates for DNA

$$q_{01}(\boldsymbol{z})=f(z_2), \quad q_{10}(\boldsymbol{z})=g(z_2), \tag{1.4}$$

and the following list of reactions associated with slow variables as shown in Table 1.1.

Reaction scheme	Propensity function	State change vector
$\mathrm{DNA_{act}}\to\mathrm{mRNA}$	$\lambda_1(z_1,z_2,\xi)=b^{-1}\xi$	$\boldsymbol{u}_1=(1,0)$
$\mathrm{mRNA}\to\emptyset$	$\lambda_2(z_1,z_2,\xi)=\gamma z_1$	$\boldsymbol{u}_2=(-1,0)$
$\mathrm{mRNA}\to\mathrm{Protein}$	$\lambda_3(z_1,z_2,\xi)=\gamma b z_1$	$\boldsymbol{u}_3=(0,1)$
$\mathrm{Protein}\to\emptyset$	$\lambda_4(z_1,z_2,\xi)=z_2$	$\boldsymbol{u}_4=(0,-1)$

TABLE 1.1. *Reaction schemes and parameters*

The infinitesimal generator of this process has the form

$$\begin{aligned}\mathcal{L}_n h(\boldsymbol{z},i)=&n\sum_{l=1}^{4}\lambda_l(\boldsymbol{z},i)\Big(h(\boldsymbol{z}+n^{-1}\boldsymbol{u}_l,i)-h(\boldsymbol{z},i)\Big)\\&+n\bigg(f(z_2)[h(\boldsymbol{z},1)-h(\boldsymbol{z},0)]+g(z_2)[h(\boldsymbol{z},0)-h(\boldsymbol{z},1)]\bigg)\end{aligned} \tag{1.5}$$

for $i=0,1$. One can obtain a mean field ODE system as

$$\frac{dz_1}{dt}=\frac{b^{-1}f(z_2)}{f(z_2)+g(z_2)}-\gamma z_1,\quad \frac{dz_2}{dt}=\gamma b z_1-z_2 \tag{1.6}$$

when n goes to infinity through the perturbation analysis for the infinitesimal generator [18, 21, 23]. With suitable choice of functions $F(Z_2)$ and $G(Z_2)$, the final mean field ODEs have two stable stationary points, and there are noise induced transitions between these two states when n is finite. To understand the robustness of the genetic switching, the biophysicists employed the WKB ansatz to the stationary distribution [1]

$$P(Z_1,Z_2)\sim\exp[-nS(z_1,z_2)] \tag{1.7}$$

and obtained a steady-state Hamilton–Jacobi equation $H(z_1,z_2,\nabla S)=0$. Mathematically, the function S resembles the role of the quasi-potential of the stochastic dynamical system [12, 22, 32], but it is not clear whether it is the case in the current stage. Another related physics approach to studying a similar switching system is to utilize the spin-boson path integral formalism in quantum field theory and then take the semi-classical approximation and adiabatic limit [19, 31]. Both approaches are difficult to be rationalized in mathematical sense. So how are we to formulate this problem in a mathematically rigorous way? To resolve this issue, we have to answer the following two fundamental questions.

(1) Question 1. What is the large deviation principle (LDP) associated with the system (1.2)? Presumably, we can obtain the Lagrangian from the large deviation analysis and then get the Hamiltonian H through the Legendre–Fenchel transform.
(2) Question 2. What is the relation between the rigorously obtained Hamiltonian H in the above question and the Hamiltonian obtained via WKB asymptotics?

The aim of this paper is to make an exploration on these two questions. To do this, we first note that the large volume limit no longer holds in the current example. Although the mRNA and protein copy numbers scale as V, we have only one DNA, which switches between the active and inactive states. This fact excludes the direct applicability of the LDP results in [26]. However, the fast switching between the two states of the DNA ensures the averaging technique still valid as shown in Equation (1.6) by taking the quasi-equilibrium limit [8, 16, 21]. We will show that the LDP analysis is also feasible by incorporating the Donsker-Varadhan type large deviations. Indeed, a similar situation has been nicely discussed by Liptser [20] and Veretennikov [29, 30] for two-scale diffusions like

$$dX_n(t)=A(X_n(t),\xi_n(t))dt+\frac{1}{\sqrt{n}}B(X_n(t),\xi_n(t))dW_t, \tag{1.8}$$

$$d\xi_n(t)=nb(\xi_n(t))dt+\sqrt{n}\sigma(\xi_n(t))dV_t. \tag{1.9}$$

The main idea of this paper is to generalize the result in [20] to our two-scale chemical kinetic processes. As we will see, although the framework is similar, we have to deal with the technicalities brought by the jump processes and the full coupling between the fast and slow variables (ξ_n is independent of X_n in Equation (1.9)).

To state the main results of this paper, let us introduce the occupation measure ν_n on $([0,T]\times\mathbb{Z}_D,\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{Z}_D))$ corresponding to ξ_n

$$\nu_n(\Delta\times\Gamma)=\int_0^T \mathbf{1}(t\in\Delta,\xi_n\in\Gamma)dt,\quad \Delta\in\mathscr{B}([0,T]),\Gamma\in\mathscr{B}(\mathbb{Z}_D), \tag{1.10}$$

where T is any fixed positive real number. Denote by $\mathbb{D}^d[0,T]$ the space of d-dimensional vector functions on $[0,T]$ whose components are right-continuous with left-hand limits, $\mathbb{M}_L[0,T]$ of finite measures $\nu=\nu(dt,i)$ on $([0,T]\times\mathbb{Z}_D,\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{Z}_D))$ which are absolutely continuous with respect to dt and have Lebesgue time marginals, i.e. we have $\nu(dt,i)=n_\nu(t,i)dt$, $n_\nu(t,i)\geq 0$ and $\sum_{i=1}^D n_\nu(t,i)=1$. The ν_n we considered always belongs to $\mathbb{M}_L[0,T]$. Take the metric $\rho^{(2)}$ on $\mathbb{M}_L[0,T]$ as the Lévy–Prohorov metric and $\rho^{(1)}$ on $\mathbb{D}^d[0,T]$ as the Skorohod metric defined as

$$\rho^{(1)}(\boldsymbol{r},\tilde{\boldsymbol{r}})=\inf_{\lambda\in\mathcal{F}}\Big\{\|\lambda\|^\circ\vee\sup_{t\in[0,T]}\|\boldsymbol{r}(t)-\tilde{\boldsymbol{r}}(\lambda(t))\|\Big\}, \tag{1.11}$$

where $\|\cdot\|$ is the Euclidean norm in the corresponding space, $\mathcal{F}$ is the collection of strictly increasing functions $\lambda(t)$ such that $\lambda(0)=0$ and $\lambda(T)=T$, and

$$\|\lambda\|^\circ:=\sup_{0\leq s<t\leq T}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right|.$$

$\mathbb{D}^d[0,T]$ and $\mathbb{M}_L[0,T]$ are complete and separable spaces with $\rho^{(1)}$ and $\rho^{(2)}$, respectively [2]. Our task is to establish the LDP for the pair $(\boldsymbol{z}_n,\nu_n)$ in metric space $(\mathbb{D}^d[0,T]\times\mathbb{M}_L[0,T],\rho^{(1)}\times\rho^{(2)})$.

This paper is organized as follows. In Section 2, we present the main large deviation theorem and give the rate functional of the whole system. By using the contraction principle and the Legendre–Fenchel transform, we get the Hamiltonian related to the slow variable $\boldsymbol{z}_n$. As a concrete application, we then study the genetic switching model and compare the difference between the rigorously obtained Hamiltonian and that obtained by WKB ansatz. In sections 3 and 4, we give the proof of the main theorem. Due to the technicalities of handling the non-negativity constraint for $\boldsymbol{r}$, we decompose the proof procedure into two steps. In Section 3, we prove the LDT theorem by relaxing the bounded domain condition to the whole space case. The upper-bound estimate is standard in some sense. However, the proof of the lower bound is technical because of the full coupling between the fast and slow variables. The resolution is based on the approximation and change-of-measure approach. The central idea is to make a piecewise linear approximation to any given path and occupation measure $(\boldsymbol{r},\nu)$ by $(\boldsymbol{y},\pi)$ at first and then construct suitable new processes $(\bar{\boldsymbol{z}}_n,\bar{\nu}_n)$ such that $\mathbb{P}-\lim_{n\to\infty}\rho^{(1)}(\bar{\boldsymbol{z}}_n,\boldsymbol{y})=0$ and $\mathbb{P}-\lim_{n\to\infty}\rho^{(2)}(\bar{\nu}_n,\pi)=0$. This turns out to be technical and one key part of the whole paper. In Section 4, we strengthen the result to the half-space case. Some details are left to Appendix A.

This paper should be considered as the companion of [19,21] for studying the rare events in genetic switching system, and it is of general interest to understand the large deviations for multiscale problems [7,8].

2. Main result and its application

2.1. Main theorem. We need the following technical assumptions for our main result.

ASSUMPTION 2.1. *Let $W:=\overline{\mathbb{R}^d_+}$. Assume the following regularity conditions for the propensity functions and jump rates hold.*

(1) (a) For each $i\in\{1,2,\dots,S\}$ and $j\in\mathbb{Z}_D$ and for all $\boldsymbol{z},\boldsymbol{x}\in W$, the Lipschitz condition holds

$$|\lambda_i(\boldsymbol{z},j)-\lambda_i(\boldsymbol{x},j)|\leq L\|\boldsymbol{z}-\boldsymbol{x}\|. \tag{2.1}$$

(b) *For each* $i \in \{1,2,\dots,S\}, j \in \mathbb{Z}_D$ *and for all* $\boldsymbol{z} \in W^\circ$, $\lambda_i(\boldsymbol{z},j) > 0$.

(c) *For each* $\boldsymbol{x} \in \partial W$ *and* $\boldsymbol{y} \in \mathcal{C}\{\boldsymbol{u}_j | \lambda_j(\boldsymbol{x}) > 0\}$, *we have* $\boldsymbol{x} + s\boldsymbol{y} \in W$ *for some* $s \in (0,\infty)$, *where* $\mathcal{C}\{\boldsymbol{u}_j\}$ *is the positive cone spanned by the vectors* $\{\boldsymbol{u}_j\}$ *defined as*

$$\mathcal{C}\{\boldsymbol{u}_j\} := \{\boldsymbol{v} | \text{there exist } \alpha_j \geq 0 \text{ such that } \boldsymbol{v} = \sum\nolimits_j \alpha_j \boldsymbol{u}_j\}. \tag{2.2}$$

(2) *For each* $i,j \in \mathbb{Z}_D$, $\log q_{ij}(\boldsymbol{z})$ *are bounded and Lipschitz continuous with respect to* $\boldsymbol{z} \in W$.

These assumptions hold in our application example in Section 2.2.

THEOREM 2.1. *Under Assumption 2.1, the family* $(\boldsymbol{z}_n, \nu_n)$ *defined by Equations* (1.1) *and* (1.10) *obeys the LDP in* $(\mathbb{D}^d[0,T] \times \mathbb{M}_L[0,T], \rho^{(1)} \times \rho^{(2)})$ *with a good rate functional* $I(\boldsymbol{r},\nu) = I_s(\boldsymbol{r},\nu) + I_f(\boldsymbol{r},\nu)$, *i.e.*

(0) $I(\boldsymbol{r},\nu)$ *values in* $[0,+\infty]$, *and its level sets are compact in* $(\mathbb{D}^d[0,T] \times \mathbb{M}_L[0,T], \rho^{(1)} \times \rho^{(2)})$,

(1) *for every closed set* $F \in \mathbb{D}^d[0,T] \times \mathbb{M}_L[0,T]$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}((\boldsymbol{z}_n,\nu_n) \in F) \leq -\inf_{(\boldsymbol{r},\nu)\in F} I(\boldsymbol{r},\nu), \tag{2.3}$$

(2) *for every open set* $G \in \mathbb{D}^d[0,T] \times \mathbb{M}_L[0,T]$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}((\boldsymbol{z}_n,\nu_n) \in G) \geq -\inf_{(\boldsymbol{r},\nu)\in G} I(\boldsymbol{r},\nu), \tag{2.4}$$

where the rate functional for the slow variables

$$I_s(\boldsymbol{r},\nu) = \begin{cases} \int_0^T L_s(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), n_\nu(t,\cdot))dt, & d\boldsymbol{r}(t) = \dot{\boldsymbol{r}}(t)dt, \\ \infty, & \textit{otherwise}, \end{cases} \tag{2.5}$$

$$L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) = \sup_{\boldsymbol{p}\in\mathbb{R}^d} (\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w})), \tag{2.6}$$

$$H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) = \sum_{i=1}^S \sum_{j=1}^D \lambda_i(\boldsymbol{z},j) w_j \left(e^{\langle \boldsymbol{p},\boldsymbol{u}_i\rangle} - 1\right), \tag{2.7}$$

and the rate functional for the fast variables

$$I_f(\boldsymbol{r},\nu) = \int_0^T S(\boldsymbol{r}(t), n_\nu(t,\cdot))dt, \tag{2.8}$$

$$S(\boldsymbol{z},\boldsymbol{w}) = \sup_{\boldsymbol{\sigma}\in\mathbb{R}^D} S(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma}), \tag{2.9}$$

$$S(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma}) = -\sum_{i,j=1}^D w_i q_{ij}(\boldsymbol{z}) \left(e^{\langle \boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle} - 1\right). \tag{2.10}$$

Here, we take the notation $\nu(dt,\cdot) = n_\nu(t,\cdot)dt$; *thus,* $n_\nu(t,\cdot)$ *is a probabilistic vector* $(n_\nu(t,1), n_\nu(t,2), \dots, n_\nu(t,D))$. $\boldsymbol{w} = (w_1, w_2, \dots, w_D)$, *and* $\langle\cdot,\cdot\rangle$ *is the inner product in* *the Euclidean space.* $\boldsymbol{e}_{ij} = \boldsymbol{e}_i - \boldsymbol{e}_j$ *and* $\{\boldsymbol{e}_i\}_{i=1}^D$ *are a canonical basis in Euclidean space* $\mathbb{R}^D$*. We take the convention that* $\boldsymbol{r}(t)$ *is absolutely continuous with respect to time when we use the notation* $d\boldsymbol{r}(t) = \dot{\boldsymbol{r}}(t)dt$*, and* S *is a function of* $(\boldsymbol{z},\boldsymbol{w})$ *(or* $(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma})$*) when we use* $S(\boldsymbol{z},\boldsymbol{w})$ *(or* $S(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma})$*) by default.*

The proof of Theorem 2.1 relies on first establishing a weaker statement based on the following stronger assumption on the whole space.

ASSUMPTION 2.2. *Regularity for the propensity functions and jump rates.*

(1) For each $i \in \{1,2,\dots,S\}$ *and* $j \in \mathbb{Z}_D$*,* $\log\lambda_i(\boldsymbol{z},j)$ *is bounded and Lipschitz continuous with respect to* $\boldsymbol{z} \in \mathbb{R}^d$*.*

(2) For each $i,j \in \mathbb{Z}_D$*,* $\log q_{ij}(\boldsymbol{z})$ *are bounded and Lipschitz continuous with respect to* $\boldsymbol{z} \in \mathbb{R}^d$*.*

This covers Assumption 2.1. Mathematically we express the boundedness of $\log\lambda_i$ and $\log q_{ij}$ as

$$\frac{1}{\Lambda} \le \lambda_i(\boldsymbol{z},j), q_{ij}(\boldsymbol{z}) \le \Lambda, \quad \Lambda > 1 \tag{2.11}$$

for any $\boldsymbol{z} \in \mathbb{R}^d$, $i \in \{1,2,\dots,S\}$ and $j \in \mathbb{Z}_D$. And in this stronger set-up we simply denote the positive cone generated by $\{\boldsymbol{u}_j\}$ as

$$\mathcal{C} := \left\{\boldsymbol{v} | \text{there exist } \alpha_j \ge 0 \text{ such that } \boldsymbol{v} = \sum\nolimits_j \alpha_j \boldsymbol{u}_j \right\}. \tag{2.12}$$

THEOREM 2.2. *The large deviation result in Theorem 2.1 holds for* $(\boldsymbol{z}_n, \nu_n) \in \mathbb{D}^d[0,T] \times \mathbb{M}_L[0,T]$ *under Assumption 2.2.*

As a straightforward application of the contraction principle, we have the following.

COROLLARY 2.1. *The slow variables* $\boldsymbol{z}_n$ *obey the LDP in* $(\mathbb{D}^d[0,T], \rho^{(1)})$ *with the rate functional*

$$I(\boldsymbol{r}) = \inf_{\nu \in \mathbb{M}_L[0,T]} (I_s(\boldsymbol{r},\nu) + I_f(\boldsymbol{r},\nu)). \tag{2.13}$$

Define the set of probabilistic transition kernels on $\mathbb{Z}_D$ as $\Delta_D = \{\boldsymbol{w} : w_1, w_2, \dots, w_D \ge 0, \sum_{i=1}^D w_i = 1\}$, where $\boldsymbol{w} = (w_1, w_2, \dots, w_D)$. We also define the reduced Lagrangian as

$$L(\boldsymbol{z},\boldsymbol{\beta}) = \inf_{\boldsymbol{w} \in \Delta_D} \{L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})\}. \tag{2.14}$$

For convenience, we will abuse the notation $n_\nu \in \mathbb{M}_L[0,T]$ and $\nu \in \mathbb{M}_L[0,T]$ in later texts.

LEMMA 2.1. *For any* $\boldsymbol{r}(\cdot)$ *which is absolutely continuous, we have*

$$\inf_{n_\nu \in \mathbb{M}_L[0,T]} \int_0^T L_s(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), n_\nu(t,\cdot)) + S(\boldsymbol{r}(t), n_\nu(t,\cdot))dt = \int_0^T L(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))dt. \tag{2.15}$$

Proof. First, let us show the measurability of the integrand on the right-hand side of Equation (2.15). By Lemma A.2, $L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})$ is convex in $\boldsymbol{w}$. So $L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})$ is continuous with respect to $\boldsymbol{w}$ in the set $\Delta_D^\circ \subset \mathbb{R}^D$ and the interior of the low dimensional boundaries of Δ_D. Choosing a countable dense subset $\{\boldsymbol{w}^k\}_{k=1}^{\infty}$ in Δ_D, we have

$$L(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t)) = \inf_{k\geq 1}\{L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),\boldsymbol{w}^k)+S(\boldsymbol{r}(t),\boldsymbol{w}^k)\} \tag{2.16}$$

for every $\boldsymbol{r}$ by the continuity condition. The measurability is a standard result with this formulation.

It is straightforward to get that

$$\inf_{n_\nu\in\mathbb{M}_L[0,T]}\int_0^T L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))+S(\boldsymbol{r}(t),n_\nu(t,\cdot))dt \geq \int_0^T L(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t))dt.$$

Now let us show the converse part. For any given $\epsilon>0$, define the sets

$$A_k=\left\{t\in[0,T]: L(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t))-\Big(L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),\boldsymbol{w}^k)+S(\boldsymbol{r}(t),\boldsymbol{w}^k)\Big)\geq -\epsilon/T\right\}$$

for $k\geq 1$. We have that A_k are measurable sets since $L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),\boldsymbol{w}^k)+S(\boldsymbol{r}(t),\boldsymbol{w}^k)$ and $L(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t))$ are both measurable functions of t. Define the measurable functions

$$F_k(t)=\begin{cases} k, & t\in A_k,\\ +\infty, & \text{otherwise}\end{cases}$$

for every $k\geq 1$ and

$$J(t)=\inf_{k\geq 1}F_k(t). \tag{2.17}$$

It is not difficult to prove that $J(t)<+\infty$ for any t and that $J(t)$ is measurable and takes values in positive integers. With these definitions, we have

$$L(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t))\geq L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),\boldsymbol{w}^{J(t)})+S(\boldsymbol{r}(t),\boldsymbol{w}^{J(t)})-\epsilon/T. \tag{2.18}$$

With $\boldsymbol{w}^{J(t)}:=\{w_1^{J(t)},w_2^{J(t)},\dots,w_D^{J(t)}\}$, define the occupation measure $\widehat{\nu}$

$$\widehat{\nu}(dt,i)=w_i^{J(t)}dt,\quad i\in\{1,2,\dots,D\}.$$

Then $\widehat{\nu}\in\mathbb{M}_L[0,T]$, $n_{\widehat{\nu}}(t,i)=w_i^{J(t)}$, and

$$\begin{aligned}\int_0^T L(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t))dt &\geq \int_0^T L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_{\widehat{\nu}}(t,\cdot))+S(\boldsymbol{r}(t),n_{\widehat{\nu}}(t,\cdot))dt-\epsilon\\ &\geq \inf_{n_\nu\in\mathbb{M}_L[0,T]}\int_0^T L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))+S(\boldsymbol{r}(t),n_\nu(t,\cdot))dt-\epsilon.\end{aligned}$$

The proof is completed. ∎

By Lemma 2.1, we have

$$\begin{aligned}I(\boldsymbol{r}) &= \inf_{\nu\in\mathbb{M}_L[0,T]}(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu))\\ &= \inf_{n_\nu\in\mathbb{M}_L[0,T]}\int_0^T L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))+S(\boldsymbol{r}(t),n_\nu(t,\cdot))dt\end{aligned}$$

$$= \int_0^T L(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))dt. \tag{2.19}$$

LEMMA 2.2. *$L(\boldsymbol{r},\boldsymbol{\beta})$ is convex in $\boldsymbol{\beta}$.*

Proof. By Lemma A.3,

$$\begin{aligned} L(\boldsymbol{z},\boldsymbol{\beta}) &= \inf_{\boldsymbol{w}\in\Delta_D} \{L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})\} \\ &= \inf_{\boldsymbol{w}\in\Delta_D} \sup_{\boldsymbol{p}\in\mathbb{R}^d} (\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})) \\ &= \sup_{\boldsymbol{p}\in\mathbb{R}^d} \inf_{\boldsymbol{w}\in\Delta_D} (\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})). \end{aligned}$$

It is easy to see that $\inf_{\boldsymbol{w}\in\Delta_D}(\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w}))$ is linear in $\boldsymbol{\beta}$; thus, $L(\boldsymbol{r},\boldsymbol{\beta})$ is convex in $\boldsymbol{\beta}$ according to Lemma A.2. ∎

It is well-known that the Lagrangian L_s does not have a closed form for the standard chemical reaction kinetic system, instead it is more convenient to investigate its dual Hamiltonian H_s by Legendre-Fenchel transform. The explicit form of the Hamiltonian is important for the numerics to study the rare events in systems biology [15]. With a similar idea, we have

$$\begin{aligned} H(\boldsymbol{z},\boldsymbol{p}) &= \sup_{\boldsymbol{\beta}\in\mathbb{R}^d} (\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - L(\boldsymbol{z},\boldsymbol{\beta})) \\ &= \sup_{\boldsymbol{\beta}\in\mathbb{R}^d} \left(\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - \inf_{\boldsymbol{w}\in\Delta_D} \{L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) + S(\boldsymbol{z},\boldsymbol{w})\}\right) \\ &= \sup_{\boldsymbol{\beta}\in\mathbb{R}^d} \sup_{\boldsymbol{w}\in\Delta_D} (\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) - S(\boldsymbol{z},\boldsymbol{w})) \\ &= \sup_{\boldsymbol{w}\in\Delta_D} \sup_{\boldsymbol{\beta}\in\mathbb{R}^d} (\langle \boldsymbol{p},\boldsymbol{\beta}\rangle - L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}) - S(\boldsymbol{z},\boldsymbol{w})) \\ &= \sup_{\boldsymbol{w}\in\Delta_D} (H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) - S(\boldsymbol{z},\boldsymbol{w})). \end{aligned} \tag{2.20}$$

A consequence about H from its definition is that H is convex with respect to $\boldsymbol{p}$ from the convexity of L and the Legendre-Fenchel transform [9]. Furthermore, if the matrix $Q = (q_{ij})_{D\times D}$ is symmetrizable, $S(\boldsymbol{z},\boldsymbol{w})$ has an explicit expression [3]

$$S(\boldsymbol{z},\boldsymbol{w}) = \frac{1}{2}\sum_i \sum_{j\neq i} \left[\sqrt{w_i q_{ij}(\boldsymbol{z})} - \sqrt{w_j q_{ji}(\boldsymbol{z})}\right]^2. \tag{2.21}$$

2.2. Application to the genetic switching model. The formula (2.20) has a nice application in the genetic switching model introduced before. In this model, we have $d = 2$, $D = 2$, and $S = 4$. By the parameters shown in Equation (1.4) and Table 1.1, we have

$$H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) = b^{-1} w_1 (e^{p_1} - 1) + A(z_1, z_2, p_1, p_2), \tag{2.22}$$

where $\boldsymbol{z} = (z_1, z_2), \boldsymbol{p} = (p_1, p_2), \boldsymbol{w} = (w_0, w_1)$ (here we utilize the notation $\boldsymbol{w} = (w_0, w_1)$ instead of $\boldsymbol{w} = (w_1, w_2)$ as mentioned in the introduction since there is only one molecule of DNA) and $A(z_1, z_2, p_1, p_2) = \gamma z_1 (e^{-p_1} - 1) + \gamma b z_1 (e^{p_2} - 1) + z_2 (e^{-p_2} - 1)$. We also have

$$S(\boldsymbol{z},\boldsymbol{w}) = \left(\sqrt{w_0 f(z_2)} - \sqrt{w_1 g(z_2)}\right)^2.$$

Applying Equation (2.20) with the constraints $w_0+w_1=1$ and $w_0,w_1\geq 0$, we obtain the final Hamiltonian

$$H(\boldsymbol{z},\boldsymbol{p})=b^{-1}s(e^{p_1}-1)-\left(\sqrt{(1-s)f(z_2)}-\sqrt{sg(z_2)}\right)^2+A(z_1,z_2,p_1,p_2), \tag{2.23}$$

where

$$s=\frac{1}{2}+\frac{s_1}{2\sqrt{s_1^2+4}},\quad s_1=\frac{b^{-1}(e^{p_1}-1)+f(z_2)-g(z_2)}{\sqrt{f(z_2)g(z_2)}}.$$

It is instructive to compare this Hamiltonian with that obtained via WKB asymptotics. In [1], another form of the Hamiltonian for this system is given via WKB asymptotics:

$$\tilde{H}(\boldsymbol{z},\boldsymbol{p})=A+g(z_2)^{-1}[A+b^{-1}(e^{p_1}-1)][f(z_2)-A], \tag{2.24}$$

where $A=A(z_1,z_2,p_1,p_2)$. The relation between the Hamiltonian $\tilde{H}$ and H is not clear so far. But one crucial difference is that H is convex with respect to the momentum variable $\boldsymbol{p}$ from the form (2.20), while $\tilde{H}$ is not. It turns out this property is crucial for the numerical computations, especially for computing the transition path in geometric minimum action method (gMAM) [15]. It is also interesting to observe that the quasi-potential $S(z_1,z_2)$ obtained from

$$H(\boldsymbol{z},\nabla S)=0\quad \text{or}\quad \tilde{H}(\boldsymbol{z},\nabla S)=0$$

is the same even though H and $\tilde{H}$ are very different [21]. It can also be verified that H is not the convex hull of $\tilde{H}$ with respect to $\boldsymbol{p}$. From the Hamilton-Jacobin theory, one may speculate that these two Hamiltonians are connected through some canonical transformation. But it is only a plausible answer which is difficult to be verified even for this concrete example.

As the large deviation results give the sharpest characterization of the considered two-scale chemical kinetic system, we can obtain the deterministic mean field ODEs and the chemical Langevin approximation for the system based on the large deviations [5], which corresponds to the law of large numbers (LLN) and the central limit theorem (CLT) for the process. Taking advantage of Equation (2.23), we get

$$\left.\frac{\partial H}{\partial p_1}\right|_{\boldsymbol{p}=\boldsymbol{0}}=\frac{b^{-1}f(z_2)}{f(z_2)+g(z_2)}-\gamma z_1,\qquad \left.\frac{\partial H}{\partial p_2}\right|_{\boldsymbol{p}=\boldsymbol{0}}=\gamma b z_1-z_2. \tag{2.25}$$

The mean field ODEs defined by

$$\frac{dz_1}{dt}=\left.\frac{\partial H}{\partial p_1}\right|_{\boldsymbol{p}=\boldsymbol{0}}\quad \text{and}\quad \frac{dz_2}{dt}=\left.\frac{\partial H}{\partial p_2}\right|_{\boldsymbol{p}=\boldsymbol{0}} \tag{2.26}$$

are exactly Equation (1.6).

Furthermore, we have

$$\begin{aligned}\left.\frac{\partial^2 H}{\partial p_1^2}\right|_{\boldsymbol{p}=\boldsymbol{0}}&=\frac{b^{-1}f(z_2)}{f(z_2)+g(z_2)}+\frac{2b^{-2}f(z_2)g(z_2)}{(f(z_2)+g(z_2))^3}+\gamma z_1,\\ \left.\frac{\partial^2 H}{\partial p_2^2}\right|_{\boldsymbol{p}=\boldsymbol{0}}&=\gamma b z_1+z_2.\end{aligned}$$

This naturally leads to the following chemical Langevin approximation:

$$\begin{aligned}\frac{dz_1}{dt} &= \left[\frac{b^{-1}f}{f+g} - \gamma z_1\right] dt + \frac{1}{\sqrt{n}}\left[\sqrt{\frac{b^{-1}f}{f+g} + \frac{2b^{-2}fg}{(f+g)^3}}\, dB_t^1 - \sqrt{\gamma z_1}\, dB_t^2\right],\\ \frac{dz_2}{dt} &= [\gamma b z_1 - z_2]\, dt + \frac{1}{\sqrt{n}}\left[\sqrt{\gamma b z_1}\, dB_t^3 - \sqrt{z_2}\, dB_t^4\right],\end{aligned} \tag{2.27}$$

where f,g are abbreviations of functions $f(z_2)$ and $g(z_2)$, and B_t^i $(i=1,\ldots,4)$ are independent standard Brownian motions. It is instructive to compare Equation (2.27) with a granted formulation by naively transplanting the Langevin approximation from the simple large volume limit [14], where the equation for z_1 reads

$$\frac{dz_1}{dt} = \left[\frac{b^{-1}f}{f+g} - \gamma z_1\right] dt + \frac{1}{\sqrt{n}}\left[\sqrt{\frac{b^{-1}f}{f+g}}\, dB_t^1 - \sqrt{\gamma z_1}\, dB_t^2\right] \tag{2.28}$$

and the equation for z_2 is the same. It is remarkable that Equation (2.27) has an additional term related to the noise dB_t^1. This additional fluctuation is induced by the fast switching of DNA states. A similar situation will also occur when we derive the chemical Langevin equations for enzymatic reactions, whereas we should take the fluctuation effect of the fast switching into consideration if the considered scaling is in our regime. However, this point does not seem to be paid much attention in previous research. Similar situation is further discussed in [19].

2.3. A useful property of the Hamiltonian H. The Hamiltonian $H(\boldsymbol{z},\boldsymbol{p})$ has some nice properties which can be utilized to simplify the computations in many cases. Assuming that $Q=(q_{ij})_{D\times D}$ is symmetrizable, according to Equation (2.20), we have

$$H(\boldsymbol{z},\boldsymbol{p}) = \sup_{\boldsymbol{w}\in\Delta_D} h(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}),$$

where

$$\begin{aligned} h(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) =& H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) - S(\boldsymbol{z},\boldsymbol{w})\\ =& \sum_{i=1}^{d}\sum_{j=1}^{D}\lambda_i(\boldsymbol{z},j)w_j(e^{\langle \boldsymbol{p},\boldsymbol{u}_i\rangle}-1) - \frac{1}{2}\sum_i\sum_{j\neq i}\left[\sqrt{w_i q_{ij}(\boldsymbol{z})} - \sqrt{w_j q_{ji}(\boldsymbol{z})}\right]^2. \end{aligned}$$

We will show that the supremum of h in Δ_D can be only taken in the interior Δ_D° of Δ_D. To do this, we first note that h is continuous in Δ_D and differentiable in Δ_D°. For any $\boldsymbol{w}_b\in\partial(\Delta_D)$, define $\boldsymbol{v}=\boldsymbol{c}_0-\boldsymbol{w}_b$ where $\boldsymbol{c}_0=(1,1,\ldots,1)/D$ is the center of Δ_D. It is easy to check that

$$\lim_{t\to 0^+}\frac{1}{t}\Big(h(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}_b+t\boldsymbol{v}) - h(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}_b)\Big) = +\infty. \tag{2.29}$$

This means that the supremum of h cannot be taken in $\partial(\Delta_D)$. Furthermore, since h is strictly concave in $\boldsymbol{w}$, there exists only one point $\boldsymbol{w}^*$ in Δ_D° such that

$$\boldsymbol{w}^* = \arg\sup_{\boldsymbol{w}\in\Delta_D} h(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}).$$

An important consequence of this fact is that we can get the derivative

$$\begin{aligned}\frac{\partial H(\boldsymbol{z},\boldsymbol{p})}{\partial \boldsymbol{p}} &= \frac{dh(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}^*(\boldsymbol{p}))}{d\boldsymbol{p}} \\ &= \frac{\partial h(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}^*)}{\partial \boldsymbol{p}} + \frac{\partial h}{\partial \boldsymbol{w}}\Big|_{\boldsymbol{w}=\boldsymbol{w}^*} \frac{\partial \boldsymbol{w}^*(\boldsymbol{p})}{\partial \boldsymbol{p}} \\ &= \frac{\partial H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}^*)}{\partial \boldsymbol{p}}.\end{aligned}$$

This is very useful to simplify the derivations when utilizing the gMAM algorithm [15] to explore the transition paths.

3. Proof of Theorem 2.2

We will mainly follow the framework in [20, 26] to make the proof. First we prove the upper bound and then the lower bound.

3.1. Upper Bound. The proof of upper bound (2.3) is standard in some sense. It is difficult to estimate the probability of $(\boldsymbol{z}_n,\nu_n)\in F$ directly. We proceed with the following steps. Firstly, we approximate $\boldsymbol{z}_n$ by $\tilde{\boldsymbol{z}}_n$, where $\tilde{\boldsymbol{z}}_n$ is an absolutely continuous path. Secondly, for a given compact set, we can get an upper bound for $(\tilde{\boldsymbol{z}}_n,\nu_n)$. Thirdly, we prove that after excluding a set of exponentially small probability, $\tilde{\boldsymbol{z}}_n$ and ν_n stay in compact sets, which means that $\tilde{\boldsymbol{z}}_n$ and ν_n are an exponentially tight sequence. And finally, we get the desired result by combing the previous steps with further estimates.

Before proceeding to the proof, let us denote by $\mathbb{C}^d[0,T]$ the collection of all continuous functions of $t\in[0,T]$ with values in $\mathbb{R}^d$. Define the sup-norm for any $\boldsymbol{r},\tilde{\boldsymbol{r}}\in\mathbb{C}^d[0,T]$

$$\rho_c^{(1)}(\boldsymbol{r},\tilde{\boldsymbol{r}}) := \sup_{0\le t\le T}\|\boldsymbol{r}(t)-\tilde{\boldsymbol{r}}(t)\|.$$

We have that $(\mathbb{C}^d[0,T],\rho_c^{(1)})$ is a Polish space. The metric $\rho_c^{(1)}$ is stronger than $\rho^{(1)}$ on $\mathbb{D}^d[0,T]$. As a consequence, every open set in $(\mathbb{D}^d[0,T],\rho^{(1)})$ is also open in $(\mathbb{D}^d[0,T],\rho_c^{(1)})$. And, if $\mathcal{K}$ is compact in $(\mathbb{C}^d[0,T],\rho_c^{(1)})$, it is also compact in $(\mathbb{D}^d[0,T],\rho_c^{(1)})$ and in $(\mathbb{D}^d[0,T],\rho^{(1)})$.

To construct the approximation of $\boldsymbol{z}_n$, we subdivide the time interval $[0,T]$ into n pieces with nodes $t_j^n = Tj/n$, $j=0,1,\dots,n$. Define the piecewise linear interpolation $\tilde{\boldsymbol{z}}_n(t)$ of $\boldsymbol{z}_n(t)$ as

$$\tilde{\boldsymbol{z}}_n(t) = (1-\gamma_j(t))\boldsymbol{z}_n(t_j^n) + \gamma_j(t)\boldsymbol{z}_n(t_{j+1}^n), \quad t\in[t_j^n,t_{j+1}^n], \tag{3.1}$$

where $\gamma_j(t) = (t-t_j)n/T\in[0,1]$.

We have the important characterization that $\tilde{\boldsymbol{z}}_n$ is exponentially equivalent to $\boldsymbol{z}_n$.

Lemma 3.1. *For each $\delta>0$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\rho^{(1)}(\boldsymbol{z}_n,\tilde{\boldsymbol{z}}_n)>\delta) = -\infty. \tag{3.2}$$

The proof of Lemma 3.1 is left to Appendix A.

For given compact sets in $\mathbb{C}^d[0,T]$, the following quasi-LDP upper bound for $(\tilde{\boldsymbol{z}}_n,\nu_n)$ holds.

Lemma 3.2. *Fix step functions $\boldsymbol{\theta}(t)\in\mathbb{R}^d$ and $\boldsymbol{\alpha}(t)\in\mathbb{R}^D$. For any $\delta>0$ and compact sets $\mathcal{K}\in\mathbb{C}^d[0,T]$ and $\mathcal{S}\in\mathbb{M}_L[0,T]$, we have*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}((\tilde{\boldsymbol{z}}_n,\nu_n)\in\mathcal{K}\times\mathcal{S}) \le -\inf_{(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}}\left(I_s^\delta(\boldsymbol{r},\nu,\boldsymbol{\theta}) + I_f^\delta(\boldsymbol{r},\nu,\boldsymbol{\alpha})\right), \tag{3.3}$$

where

$$I_s^\delta(\boldsymbol{r},\nu,\boldsymbol{\theta}) = \begin{cases} \int_0^T L_s^\delta(\boldsymbol{r}(t),\dot{\boldsymbol{r}},n_\nu(t,\cdot),\boldsymbol{\theta}(t))dt, & d\boldsymbol{r}(t)=\dot{\boldsymbol{r}}(t)dt, \\ \infty, & \textit{otherwise}, \end{cases} \tag{3.4}$$

$$L_s^\delta(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p}) = \langle \boldsymbol{\beta},\boldsymbol{p}\rangle - H_s^\delta(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}), \tag{3.5}$$

$$H_s^\delta(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w}) = \sup_{|\boldsymbol{x}-\boldsymbol{z}|<\delta} H_s(\boldsymbol{x},\boldsymbol{p},\boldsymbol{w}), \tag{3.6}$$

and

$$I_f^\delta(\boldsymbol{r},\nu,\boldsymbol{\alpha}) = \int_0^T S^\delta(\boldsymbol{r}(t),n_\nu(t,\cdot),\boldsymbol{\alpha}(t))dt, \tag{3.7}$$

$$S^\delta(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma}) = -\sup_{|\boldsymbol{x}-\boldsymbol{z}|<\delta} \sum_{i,j=1}^D w_i q_{ij}(\boldsymbol{z})\left(e^{\langle \boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle}-1\right). \tag{3.8}$$

Before proceeding to the proof, we remark that $H_s^\delta(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w})$ and $L_s^\delta(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p})$ are monotonically increasing and decreasing functions of δ, respectively. $S^\delta(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma})$ is a monotonically decreasing function of δ. Correspondingly, $I_s^\delta(\boldsymbol{r},\nu,\boldsymbol{\theta})$ and $I_f^\delta(\boldsymbol{r},\nu,\boldsymbol{\alpha})$ are decreasing functionals of δ.

Proof. We only need to consider absolutely continuous functions $\boldsymbol{r}$ on the right-hand side of Equation (3.3) since $I_s^\delta(\boldsymbol{r},\nu,\boldsymbol{\theta})+I_f^\delta(\boldsymbol{r},\nu,\boldsymbol{\alpha})=\infty$ otherwise. For any $\boldsymbol{r}$ and ν, define the sum

$$\begin{aligned} J_n(\boldsymbol{r},\boldsymbol{\theta},\nu,\boldsymbol{\alpha}) = \sum_{j=0}^{n-1} \Bigg(& \left\langle \boldsymbol{r}(t_{j+1}^n)-\boldsymbol{r}(t_j^n),\boldsymbol{\theta}(t_j^n)\right\rangle \\ - \int_{t_j^n}^{t_{j+1}^n} & H_s^\delta\left(\boldsymbol{r}(t_j^n),\boldsymbol{\theta}(t_j^n),n_\nu(t,\cdot)\right)dt + \int_{t_j^n}^{t_{j+1}^n} S^\delta(\boldsymbol{r}(t_j^n),n_\nu(t,\cdot),\boldsymbol{\alpha}(t_j^n))dt \Bigg). \end{aligned} \tag{3.9}$$

By Corollary A.2 in Appendix A, we have

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{E}\exp(nJ_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha})) \le 0. \tag{3.10}$$

For $(\tilde{\boldsymbol{z}}_n,\nu_n)\in\mathcal{K}\times\mathcal{S}$, it is obvious that

$$J_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha}) - \inf_{(\boldsymbol{r},v)\in\mathcal{K}\times\mathcal{S}} J_n(\boldsymbol{r},\boldsymbol{\theta},\nu,\boldsymbol{\alpha}) \ge 0. \tag{3.11}$$

So we have

$$\exp\left(J_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha}) - \inf_{(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}} J_n(\boldsymbol{r},\boldsymbol{\theta},\nu,\boldsymbol{\alpha})\right) \ge 1$$

and

$$\mathbb{P}((\tilde{\boldsymbol{z}}_n,\nu_n)\in\mathcal{K}\times\mathcal{S}) \le \mathbb{E}\exp\left\{n\Big[J_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha}) - \inf_{(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}} J_n(\boldsymbol{r},\boldsymbol{\theta},\nu,\boldsymbol{\alpha})\Big]\right\}.$$

Combining this with Equation (3.10), we get

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}((\tilde{\boldsymbol{z}}_n,\nu_n)\in\mathcal{K}\times\mathcal{S}) \le -\liminf_{n\to\infty}\left(\inf_{(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}} J_n(\boldsymbol{r},\boldsymbol{\theta},\nu,\boldsymbol{\alpha})\right). \tag{3.12}$$

We now represent the sum on the right-hand side of Equation (3.12) as an integral. Since $\mathcal{K}$ is compact, the absolutely continuous functions $\boldsymbol{r}\in\mathcal{K}$ are thus uniformly bounded. Let V be a compact set in $\mathbb{R}^d$ such that

$$\big\{\boldsymbol{z}:\boldsymbol{z}=\boldsymbol{r}(t) \text{ for some } \boldsymbol{r}\in\mathcal{K} \text{ and } t\in[0,T]\big\}\subset V.$$

For step function $\boldsymbol{\theta}$, let us investigate an interval in which $\boldsymbol{\theta}$ takes constant value $\boldsymbol{\theta}_0$ – say, the interval $[0,\tau]$ – without loss of generality. Then

$$\sum_{j=0}^{n-1}\chi_{\{t_{j+1}^n\leqslant\tau\}}\big\langle \boldsymbol{r}(t_{j+1}^n)-\boldsymbol{r}(t_j^n),\boldsymbol{\theta}_0\big\rangle=\int_0^\tau\langle\dot{\boldsymbol{r}},\boldsymbol{\theta}_0\rangle dt+\epsilon_n,$$

where the error ϵ_n takes into account the fact that τ may not match any of t_j^n. It goes to zero uniformly for $\boldsymbol{r}\in\mathcal{K}$ when n goes to infinity from the bound

$$|\epsilon_n|\le\frac{2T}{n}|\boldsymbol{\theta}_0|\sup_{\boldsymbol{z}\in V}|\boldsymbol{z}|.$$

Now $H_s^\delta(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w})$ is continuous in $\boldsymbol{z}$, $\boldsymbol{p}$, and $\boldsymbol{w}$ from the continuity of H_s on $\boldsymbol{x}$, $\boldsymbol{p}$, and $\boldsymbol{w}$ and the boundedness of λ_i, $\boldsymbol{\theta}$, and $n_\nu(t,\cdot)$ in the current setting. So we have

$$\big|H_s^\delta\big(\boldsymbol{r}(t_j^n),\boldsymbol{\theta}(t_j^n),n_\nu(t,\cdot)\big)-H_s^\delta\big(\boldsymbol{r}(t),\boldsymbol{\theta}(t_j^n),n_\nu(t,\cdot)\big)\big|,\quad t_j^n\le t\le t_{j+1}^n$$

goes to zero uniformly in j for $\boldsymbol{r}\in\mathcal{K}$ and $\nu\in\mathcal{S}$ by equicontinuity. Therefore,

$$\sum_{j=0}^{n-1}\chi_{\{t_{j+1}^n\leqslant\tau\}}\int_{t_j^n}^{t_{j+1}^n}H_s^\delta\big(\boldsymbol{r}(t_j^n),\boldsymbol{\theta}(t_j^n),n_\nu(t,\cdot)\big)\,dt=\int_0^\tau H_s^\delta\big(\boldsymbol{r}(t),\boldsymbol{\theta}(t_j^n),n_\nu(t,\cdot)\big)\,dt+\epsilon_n,$$

with ϵ_n converging to zero uniformly in $(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}$.

Similarly, we can estimate for the part $S^\delta(\boldsymbol{r}(t_j^n),n_\nu(t,\cdot),\boldsymbol{\alpha}(t_j^n))$ and repeat the argument on the finite number of intervals on which $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}$ are constants. Thanks to the uniformity in $(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}$, we obtain

$$\liminf_{n\to\infty}\left(\inf_{(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}}J_n(\boldsymbol{r},\boldsymbol{\theta},\nu,\boldsymbol{\alpha})\right)=\inf_{(\boldsymbol{r},\nu)\in\mathcal{K}\times\mathcal{S}}(I_s^\delta(\boldsymbol{r},\nu,\boldsymbol{\theta})+I_f^\delta(\boldsymbol{r},\nu,\boldsymbol{\alpha})).$$

Together with Equation (3.12), the proof is completed. □

Next we show the exponential tightness of the sequence $(\tilde{\boldsymbol{z}}_n,\nu_n)$. Define the modulus of continuity of a continuous function $\boldsymbol{z}$ as

$$V_\delta(\boldsymbol{z})=\sup\big\{\|\boldsymbol{z}(t)-\boldsymbol{z}(s)\|:0\le s\le t\le T,|t-s|<\delta\big\}\tag{3.13}$$

and the set

$$\mathcal{K}(M)=\bigcap_{m=M}^{\infty}\left\{\boldsymbol{z}\in\mathbb{C}^d[0,T]:\boldsymbol{z}(0)=\boldsymbol{z}^0,V_{2^{-m}}(\boldsymbol{z})\le\frac{1}{\log m}\right\}\tag{3.14}$$

for any fixed $M\in\mathbb{N}$.

Lemma 3.3 (Exponential tightness for $\tilde{\boldsymbol{z}}_n$). *For each $B>0$, there is a compact set $\mathcal{K}\subset\mathbb{C}^d[0,T]$ such that*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\tilde{\boldsymbol{z}}_n\notin\mathcal{K})\le-B.$$

Proof. For any fixed $M\in\mathbb{N}$, it is not difficult to see that the set $\mathcal{K}(M)$ is closed and the functions in $\mathcal{K}(M)$ are equicontinuous. Thus, $\mathcal{K}(M)$ is compact by the Arzela-Ascoli theorem. If $2^{-m}<T/n$, we have

$$V_{2^{-m-1}}(\tilde{\boldsymbol{z}}_n)=\frac{1}{2}V_{2^{-m}}(\tilde{\boldsymbol{z}}_n)$$

since $\tilde{\boldsymbol{z}}_n$ is piecewise linear. Therefore, to check whether $\tilde{\boldsymbol{z}}_n$ is in $\mathcal{K}(M)$, we only need to consider a finite intersection, for values of m up to

$$M(n)=\max\left\{M,\left\lceil\frac{\log(n/T)}{\log 2}\right\rceil\right\}.$$

Using Corollary A.1 in Appendix A, we have for any n with $M(n)>M$,

$$\begin{aligned}\mathbb{P}(\tilde{\boldsymbol{z}}_n\notin\mathcal{K}(M))&\leq\sum_{m=M}^{M(n)}\mathbb{P}\left(V_{2^{-m}}(\tilde{\boldsymbol{z}}_n)>\frac{1}{\log m}\right)\\&\leq\sum_{m=M}^{M(n)}\sum_{j=0}^{n-1}\mathbb{P}\left(\sup_{0\leq t\leq 2^{-m}}|\boldsymbol{z}_n(t_j^n+t)-\boldsymbol{z}_n(t_j^n)|>\frac{1}{\log M}\right)\\&\leq nM(n)\cdot 2d\exp\left(-n\frac{c_1}{\log M}\log\left(\frac{2^Mc_2}{\log M}\right)\right)\end{aligned}$$

for positive constants c_1 and c_2. Thus

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\tilde{\boldsymbol{z}}_n\notin\mathcal{K}(M))\leq -c\frac{M}{\log M}$$

for some positive constant c when $M\gg 1$. ∎

LEMMA 3.4. *The measure space $\mathbb{M}_L[0,T]$ is compact.*

Proof. Since $[0,T]\times\{1,2,\dots,D\}$ is compact, $\mathbb{M}_L[0,T]$ is tight. By Prohorov's theorem, $\mathbb{M}_L[0,T]$ is relatively compact. Let ν be the limit of any converging sequence $\{\nu_m\}$ in $\mathbb{M}_L[0,T]$. Since $\sum_{i=1}^D\nu_m(dt,i)=dt$ for all m, we have $\sum_{i=1}^D\nu(dt,i)=dt$, and thus $\nu(dt,i)\ll dt$. So ν also belongs to $\mathbb{M}_L[0,T]$. This proves that $\mathbb{M}_L[0,T]$ is compact. ∎

The straightforward consequence of Lemma 3.4 is that ν_n is also exponentially tight.

Define the quasi-rate functionals for slow and fast variables corresponding to I_s and I_f in Theorem 2.1

$$I_s^\delta(\boldsymbol{r},\nu)=\begin{cases}\int_0^T L_s^\delta(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))dt, & d\boldsymbol{r}(t)=\dot{\boldsymbol{r}}(t)dt,\\ \infty, & \text{otherwise},\end{cases}\tag{3.15}$$

$$L_s^\delta(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})=\sup_{\boldsymbol{p}\in\mathbb{R}^d}L_s^\delta(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p}),\tag{3.16}$$

and

$$I_f^\delta(\boldsymbol{r},\nu)=\int_0^T S^\delta(\boldsymbol{r}(t),n_\nu(t,\cdot))dt,\tag{3.17}$$

$$S^{\delta}(\boldsymbol{z},\boldsymbol{w}) = \sup_{\boldsymbol{\sigma}\in\mathbb{R}^D} S^{\delta}(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma}). \tag{3.18}$$

The definitions of $L_s^{\delta}(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p})$ and $S^{\delta}(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma})$ are referred to in Equations (3.5) and (3.8). We have the following approximation lemmas.

LEMMA 3.5. *For any $\epsilon>0$, the absolutely continuous function $\boldsymbol{r}\in\mathbb{C}^d[0,T]$ and $\nu\in\mathbb{M}_L[0,T]$, there exists neighborhood $N_{\boldsymbol{r},\nu}\in\mathbb{C}^d[0,T]\times\mathbb{M}_L[0,T]$ of $(\boldsymbol{r},\nu)$, step functions $\boldsymbol{\theta}_{\boldsymbol{r}\nu}\subset\mathbb{R}^d$ and $\boldsymbol{\alpha}_{\boldsymbol{r}\nu}\subset\mathbb{R}^D$, such that for any $(\boldsymbol{q},\mu)\in N_{\boldsymbol{r},\nu}$, we have*

$$I_s^{\delta}(\boldsymbol{q},\mu,\boldsymbol{\theta}_{\boldsymbol{r}\nu}) + I_f^{\delta}(\boldsymbol{q},\mu,\boldsymbol{\alpha}_{\boldsymbol{r}\nu}) \ge I_s^{\delta}(\boldsymbol{r},\nu) + I_f^{\delta}(\boldsymbol{r},\nu) - \epsilon.$$

LEMMA 3.6. *For any pair $(\boldsymbol{r},\nu)\in\mathbb{C}^d[0,T]\times\mathbb{M}_L[0,T]$ and $M_0>0$, if $\boldsymbol{r}$ is not absolutely continuous, there exists neighborhood $N_{\boldsymbol{r},\nu}\in\mathbb{C}^d[0,T]\times\mathbb{M}_L[0,T]$ of $(\boldsymbol{r},\nu)$ and step functions $\boldsymbol{\theta}_{\boldsymbol{r}\nu}\in\mathbb{R}^d$ and $\boldsymbol{\alpha}_{\boldsymbol{r}\nu}\in\mathbb{R}^D$, such that for any $(\boldsymbol{q},\mu)\in N_{\boldsymbol{r},\nu}$, we have*

$$I_s^{\delta}(\boldsymbol{q},\mu,\boldsymbol{\theta}_{\boldsymbol{r}\nu}) + I_f^{\delta}(\boldsymbol{q},\mu,\boldsymbol{\alpha}_{\boldsymbol{r}\nu}) \ge M_0.$$

Lemmas 3.5 and 3.6 are direct consequences of lemmas A.8 and A.9 in Appendix A.

Simply denote the product metric $\rho^{(1)}\times\rho^{(2)}$ on $\mathbb{D}^d[0,T]\times\mathbb{M}_L[0,T]$ as $d(\cdot,\cdot)$ and define the sets

$$\Phi(K) = \left\{(\boldsymbol{r},\nu)\in\mathbb{D}^d[0,T]\times\mathbb{M}_L[0,T] : I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)\le K\right\} \tag{3.19}$$

and

$$\Phi^{\delta}(K) = \left\{(\boldsymbol{r},\nu)\in\mathbb{D}^d[0,T]\times\mathbb{M}_L[0,T] : I_s^{\delta}(\boldsymbol{r},\nu)+I_f^{\delta}(\boldsymbol{r},\nu)\le K\right\}. \tag{3.20}$$

We have the following characterization for $\Phi(K)$ and $\Phi^{\delta}(K)$.

LEMMA 3.7. *For any $K>0$, the level sets $\Phi(K)$ and $\Phi^{\delta}(K)$ defined in Equations (3.19) and (3.20) are compact sets.*

Proof. By Lemma 3.4, $\mathbb{M}_L[0,T]$ is a compact set. By Lemma A.7, the functions $\boldsymbol{r}\in\Phi(K)$ are equicontinuous. Combining this with the fact that $\boldsymbol{r}(0)=\boldsymbol{z}^0$, we have that $\Phi(K)$ is pre-compact. By Lemma A.8, $I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)$ is lower semicontinuous. Consequently, $\Phi(K)$ is closed and thus compact. The proof for $\Phi^{\delta}(K)$ is similar. □

PROPOSITION 3.1. *For each $K>0$, $\delta>0$, and $\epsilon>0$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(d\left((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi^{\delta}(K)\right)>\epsilon\right)\le-(K-\epsilon).$$

Proof. From the exponential tightness, we can find a compact set $\mathcal{K}^N\in\mathbb{C}^d[0,T]$ for each $N>0$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\tilde{\boldsymbol{z}}_n\notin\mathcal{K}^N)\le -N.$$

Define the set

$$\mathcal{K}^{N,\epsilon} = \overline{\{(\boldsymbol{r},\nu)\in\mathbb{C}^d[0,T]\times\mathbb{M}_L[0,T] : d((\boldsymbol{r},\nu),\Phi^{\delta}(K))>\epsilon\}}\cap(\mathcal{K}^N\times\mathbb{M}_L[0,T]).$$

For any $(\boldsymbol{r},\nu)\in\mathcal{K}^{N,\epsilon}$, we can find the neighborhood $N_{\boldsymbol{r},\nu}$ either satisfying Lemma 3.5, if $\boldsymbol{r}$ is absolutely continuous, or satisfying Lemma 3.6 if $\boldsymbol{r}$ is not absolutely continuous. This forms a covering of $\mathcal{K}^{N,\epsilon}$. By compactness, we can choose a finite subcover $\{N_{\boldsymbol{r}_i,\nu_j}\}_{i,j}$ for $\mathcal{K}^{N,\epsilon}$. Define

$$\mathcal{K}_{ij}=\overline{N_{\boldsymbol{r}_i,\nu_j}\cap\mathcal{K}^{N,\epsilon}}.$$

Applying lemmas 3.2, 3.5, and 3.6 and letting M_0 in Lemma 3.6 be larger than K, we have for any i,j

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}((\tilde{\boldsymbol{z}}_n,\nu_n)\in\mathcal{K}_{ij})\leq-(K-\epsilon).$$

Then we have

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(d\left((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi^\delta(K)\right)>\epsilon\right)\\ &\leq\limsup_{n\to\infty}\frac{1}{n}\log\left[\mathbb{P}(\tilde{\boldsymbol{z}}_n\notin\mathcal{K}^N)+\sum_{i,j}\mathbb{P}((\tilde{\boldsymbol{z}}_n,\nu_n)\in\mathcal{K}_{ij})\right]\\ &\leq-\min\{N,K-\epsilon\}.\end{aligned}$$

Choosing N large enough, we complete the proof. ∎

We are now ready to establish the upper bound.

LEMMA 3.8. *Given $K>0$ and $\epsilon>0$, there exist $\delta>0$ such that*

$$\Phi^\delta(K-\epsilon)\subset\{(\boldsymbol{r},\nu):d((\boldsymbol{r},\nu),\Phi(K))\leq\epsilon\}.$$

Proof. Prove by contradiction. If the claim is false, we can choose

$$\delta_i\downarrow0,\qquad(\boldsymbol{r}_i,\nu_i)\in\Phi^{\delta_i}(K-\epsilon),\quad i=1,2,\ldots$$

such that

$$d((\boldsymbol{r}_i,\nu_i),\Phi(K))\geq\epsilon,\quad\forall i.\tag{3.21}$$

By the definition of $I_s^\delta(\boldsymbol{r},\nu)$ and $I_f^\delta(\boldsymbol{r},\nu)$, we have the monotonicity $I_s^\delta(\boldsymbol{r},\nu)\leq I_s^{\delta'}(\boldsymbol{r},\nu)$ and $I_f^\delta(\boldsymbol{r},\nu)\leq I_f^{\delta'}(\boldsymbol{r},\nu)$ when $\delta\geq\delta'\geq0$. Thus, the sets $\Phi^{\delta_i}(K-\epsilon)$ are monotonically decreasing as $\delta_i\downarrow0$, and $(\boldsymbol{r}_i,\nu_i)$ are contained in the set $\Phi^{\delta_1}(K-\epsilon)$ which is compact by Lemma 3.7. So there exists a subsequence converging to $(\boldsymbol{r}_0,\nu_0)$. With Lemma A.8 in Appendix A we have for each j

$$\begin{aligned}I_s^{\delta_j}(\boldsymbol{r}_0,\nu_0)+I_f^{\delta_j}(\boldsymbol{r}_0,\nu_0)&\leq\liminf_{i\to\infty}\left(I_s^{\delta_j}(\boldsymbol{r}_i,\nu_i)+I_f^{\delta_j}(\boldsymbol{r}_i,\nu_i)\right)\\ &\leq\liminf_{i\to\infty}\left(I_s^{\delta_i}(\boldsymbol{r}_i,\nu_i)+I_f^{\delta_i}(\boldsymbol{r}_i,\nu_i)\right)\\ &\leq K-\epsilon.\end{aligned}$$

The monotone convergence theorem gives

$$I_s(\boldsymbol{r}_0,\nu_0)+I_f(\boldsymbol{r}_0,\nu_0)=\lim_{j\to\infty}I_s^{\delta_j}(\boldsymbol{r}_0,\nu_0)+I_f^{\delta_j}(\boldsymbol{r}_0,\nu_0)\leq K-\epsilon.$$

So $(\boldsymbol{r}_0,\nu_0)\in\Phi(K)$. For sufficiently large i, $d((\boldsymbol{r}_0,\nu_0),(\boldsymbol{r}_i,\nu_i))\leq\epsilon$. This contradicts Equation (3.21). ∎

THEOREM 3.1. *For each closed set $F\subset\mathbb{D}^d[0,T]\times\mathbb{M}_L[0,T]$,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}((\boldsymbol{z}_n,\nu_n)\in F)\le -\inf_{(\boldsymbol{r},\nu)\in F}(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)).$$

Proof. Suppose $\inf_{(\boldsymbol{r},\nu)\in F}(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu))=K<\infty$. Since F and $\Phi(K-\epsilon)$ are both closed sets, we assume the distance between them is $\eta_0>0$. For any $\eta\le\eta_0$,

$$\begin{aligned}&\mathbb{P}((\boldsymbol{z}_n,\nu_n)\in F)\\ &\le\mathbb{P}\left(d((\tilde{\boldsymbol{z}}_n,\nu_n),F)\le\frac{\eta}{2}\right)+\mathbb{P}\left(d(\tilde{\boldsymbol{z}}_n,\nu_n),(\boldsymbol{z}_n,\nu_n))\ge\frac{\eta}{2}\right)\\ &\le\mathbb{P}\left(d((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi(K-\epsilon))\ge\frac{\eta}{2}\right)+\mathbb{P}\left(\rho^{(1)}((\tilde{\boldsymbol{z}}_n,\boldsymbol{z}_n)\ge\frac{\eta}{2}\right).\end{aligned}\tag{3.22}$$

By Lemma 3.8, we can choose δ and η small enough so that

$$d\big((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi(K-\epsilon)\big)\ge\frac{\eta}{2}\quad\text{implies}\quad d\left((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi^\delta(K-\epsilon-\eta/4)\right)\ge\frac{\eta}{4}.$$

From Proposition 3.1, we have

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(d((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi(K-\epsilon))\ge\frac{\eta}{2}\right)\\ &\le\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(d\left((\tilde{\boldsymbol{z}}_n,\nu_n),\Phi^\delta(K-\epsilon-\eta/4)\right)\ge\frac{\eta}{4}\right)\\ &\le-(K-\epsilon-\eta/2).\end{aligned}\tag{3.23}$$

Combining Equations (3.22) and (3.23) and Lemma 3.1 for $\delta=\eta/4$, we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}((\boldsymbol{z}_n,\nu_n)\in F)\le-(K-\epsilon-\eta/2).$$

The case for

$$\inf_{(\boldsymbol{r},\nu)\in F}(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu))=\infty$$

can be established similarly by choosing K arbitrarily large. ∎

3.2. Lower bound. The proof of the lower bound is based on the change of measure formula. From [5], it suffices to prove that for any $(\boldsymbol{r},\nu)\in\mathbb{D}^d[0,T]\times\mathbb{M}_L[0,T]$ and arbitrarily small $\epsilon>0$, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_\epsilon(\nu))\ge-(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)),\tag{3.24}$$

where $N_\epsilon(\boldsymbol{r})$ is the ϵ-neighborhood of $\boldsymbol{r}$ in $\mathbb{D}^d[0,T]$ with metric $\rho^{(1)}$ and where $N_\epsilon(\nu)$ is the ϵ-neighborhood of ν in $\mathbb{M}_L[0,T]$ with metric $\rho^{(2)}$. For given $\boldsymbol{r}\in\mathbb{D}^d[0,T]$ and $\nu\in\mathbb{M}_L[0,T]$, if $\boldsymbol{r}$ is not absolutely continuous, $I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)=\infty$, thus nothing needs to be proved. Below we will exclude this case. For convenience, we further assume that $n_\nu(t,i)$ is continuous in t, and the case that $n_\nu(t,i)$ is not continuous will be discussed in Theorem 3.2 in this section. To prove the lower bound, we perform the following steps. First, we approximate $\boldsymbol{r}$ by a piecewise linear path $\boldsymbol{y}$ and the occupation measure ν by $\pi\in\mathbb{M}_L[0,T]$ with $n_\pi(t,\cdot)$ piecewise constant in t. Secondly, we construct new processes $\bar{\boldsymbol{z}}_n$ and $\bar{\xi}_n$ with occupation measure $\bar{\nu}_n$ such that

$$\mathbb{P}-\lim_{n\to\infty}\rho_c^{(1)}(\bar{\boldsymbol{z}}_n,\boldsymbol{y})=0,\quad\mathbb{P}-\lim_{n\to\infty}\rho^{(2)}(\bar{\nu}_n,\pi)=0,\tag{3.25}$$

where the notation $\mathbb{P}-\lim$ means the convergence in probability. Moreover, we ask that $\bar{\boldsymbol{z}}_n$ and the jump rates of $\bar{\xi}_n$ satisfy the conditions required by lemmas 3.9 and 3.10. Finally, based on the change of measure formula related to $(\boldsymbol{z}_n,\xi_n)$ and $(\bar{\boldsymbol{z}}_n,\bar{\xi}_n)$, we get the limit, and the proof is then finished.

As promised in the above procedure, we approximate $\boldsymbol{r}$ by a path $\boldsymbol{y}$ first. For a given J, define $\Delta=T/J$ and let $t_m=m\Delta$. On each interval $[t_m,t_{m+1}]$, define $\Delta\boldsymbol{r}_m=\boldsymbol{r}(t_{m+1})-\boldsymbol{r}(t_m)$. Take $\boldsymbol{\mu}^m=\{\mu_i^m,i=1,\dots,S\}$ so as to satisfy

$$\sum_{i=1}^{S}\mu_i^m\boldsymbol{u}_i=\frac{\Delta\boldsymbol{r}_m}{\Delta}\quad\text{and}\quad\mu_i^m\geq 0.\tag{3.26}$$

If $\Delta\boldsymbol{r}_m$ are in the positive cone generated by the $\{\boldsymbol{u}_i\}$ for all m, such a choice of $\boldsymbol{\mu}^m$ is possible. If at least one of $\Delta\boldsymbol{r}_m$ is not in the positive cone generated by the $\{\boldsymbol{u}_i\}$, it is easy to check that, for all $\nu\in\mathbb{M}[0,T]$, $I_s(\boldsymbol{r},\nu)=+\infty$ (see the Remark of [26, Lemma 5.21]) and nothing needs to be proved.

Now we construct the piecewise linear interpolation $\boldsymbol{y}$ of $\boldsymbol{r}$ such that $\boldsymbol{y}(t_0)=\boldsymbol{r}(t_0)$ and in each time interval $[t_m,t_{m+1}]$

$$\frac{d}{dt}\boldsymbol{y}(t)=\sum_{i=1}^{S}\mu_i^m\boldsymbol{u}_i.\tag{3.27}$$

Thus, $\boldsymbol{y}(t_m)=\boldsymbol{r}(t_m)$ for each m. For any $\epsilon>0$, we can choose J large enough such that

$$\rho_c^{(1)}(\boldsymbol{y},\boldsymbol{r})<\epsilon/4.$$

Define the sets

$$\mathcal{S}=\Big\{(\eta,\psi)\big|\eta=(\eta_{ij})_{D\times D},\ \eta_{ij}>0;\ \psi\in\Delta_D;\sum_{i=1}^{D}\psi_i\sum_{j=1}^{D}\eta_{ij}\boldsymbol{e}_{ij}=\boldsymbol{0}\Big\}\tag{3.28}$$

and

$$\mathcal{K}_{\boldsymbol{\beta}}=\Big\{\boldsymbol{\mu}\in\mathbb{R}^S:\mu_i\geq 0,\sum_{i=1}^{S}\mu_i\boldsymbol{u}_i=\boldsymbol{\beta}\Big\}.\tag{3.29}$$

We remark that the sets $\mathcal{S}$ and $\mathcal{K}_{\boldsymbol{\beta}}$ here have nothing to do with the definitions in the proof of upper bound.

LEMMA 3.9. *For any $\epsilon>0$ and large enough J, there exists a further subdivision of time interval $[t_m,t_{m+1}]$ for each $m\in\{0,1,\dots,J-1\}$ (i.e. $t_m=t_{m0}<t_{m1}<\cdots<t_{mK_m}=t_{m+1}$) and related $(\eta^{mk},\psi^{mk})\in\mathcal{S}$ ($m=0,1,\dots,J-1;k=0,1,\dots,K_m-1$), such that*

$$\sum_{m=0}^{J-1}\sum_{k=0}^{K_m-1}\int_{t_{mk}}^{t_{m,k+1}}\sum_{i=1}^{D}\psi_i^{mk}\sum_{j=1}^{D}\Big(\eta_{ij}^{mk}\log\frac{\eta_{ij}^{mk}}{q_{ij}(\boldsymbol{y}(t))}+q_{ij}(\boldsymbol{y}(t))-\eta_{ij}^{mk}\Big)dt$$
$$\leq I_f(\boldsymbol{r},\nu)+\epsilon.$$

and

$$\|\psi^{mk}-n_\nu(t,\cdot)\|<\epsilon/(4DT)\tag{3.30}$$

for all $t\in[t_{mk},t_{m,k+1})$, $k=0,1,\dots,K_m-1$ and $m=0,1,\dots,J-1$.

The proof of Lemma 3.9 can be found in Appendix A.

We then define the measure $\pi\in\mathbb{M}_L[0,T]$ such that $\pi(dt,i)=n_\pi(t,i)dt$ and

$$n_\pi(t,i):=\psi_i^{mk},\quad t\in[t_{mk},t_{m,k+1})$$

for $m=0,1,\dots,J-1$ and $k=0,1,\dots,K_m-1$. With this choice $n_\pi(t,\cdot)$ is piecewise constant and

$$\rho^{(2)}(\pi,\nu)<\epsilon/4.$$

We take the frequently used notation λ_i^π in later text as the expectation of λ_i with respect to the distribution n_π

$$\lambda_i^\pi(\boldsymbol{y}(s))=\sum_{j=1}^D\lambda_i(\boldsymbol{y}(s),j)n_\pi(s,j).\tag{3.31}$$

LEMMA 3.10. *For any $\epsilon>0$ and large enough J, define $\boldsymbol{\beta}_m=\Delta r_m/\Delta$, then there exists $\boldsymbol{\mu}^m\in K_{\boldsymbol{\beta}_m}$ such that*

$$\sum_{m=0}^{J-1}\int_{t_m}^{t_{m+1}}\sum_{i=1}^S\left(\lambda_i^\pi(\boldsymbol{y}(t))-\mu_i^m+\mu_i^m\log\frac{\mu_i^m}{\lambda_i^\pi(\boldsymbol{y}(t))}\right)dt\le I_s(\boldsymbol{r},\nu)+\epsilon.$$

The proof of Lemma 3.10 can be found in Appendix A.

With the constructed matrices $\{\eta^{mk}\}$ in Lemma 3.9, we define the process $\bar{\xi}_n$ with jump rate $n\eta_{ij}(t)$ where $\eta_{ij}(t)=\eta_{ij}^{mk}$, $t\in[t_{mk},t_{m,k+1})$. Similarly, we take $\boldsymbol{\mu}^m$ constructed from Lemma 3.10 and define $\bar{\boldsymbol{z}}_n$ with jump rate

$$n\mu_i(t)\frac{\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))}$$

for its ith component, where $\mu_i(t)$ is piecewise constant and $\mu_i(t)=\mu_i^m$ for $t\in[t_m,t_{m+1})$.

We have the following convergence result for the constructed approximations for π and $\boldsymbol{y}$.

LEMMA 3.11. *Convergence of the approximation $\bar{\nu}_n$*

$$\mathbb{P}-\lim_{n\to\infty}\rho^{(2)}(\bar{\nu}_n,\pi)=0.$$

LEMMA 3.12. *Convergence of the approximation $\bar{\boldsymbol{z}}_n$*

$$\mathbb{P}-\lim_{n\to\infty}\rho_c^{(1)}(\bar{\boldsymbol{z}}_n,\boldsymbol{y})=0.$$

The proof of lemmas 3.11 and 3.12 are given in Appendix A.

As we have finished the construction of $\bar{\boldsymbol{z}}_n$ and $\bar{\xi}_n$, we now perform the change of measure. Denote by $\mathbb{Q}_n$ and $\bar{\mathbb{Q}}_n$ the distributions of $(\boldsymbol{z}_n(t),\xi_n(t))_{t\le T}$ and $(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))_{t\le T}$, respectively. We have

$$\begin{aligned}&\frac{d\mathbb{Q}_n}{d\bar{\mathbb{Q}}_n}(\bar{\boldsymbol{z}}_n,\bar{\xi}_n)\\&=\exp\left\{-\int_0^T n\sum_{i=1}^d\left(\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))-\mu_i(t)\frac{\lambda_i(\bar{z}_n(t),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))}\right)dt\right.\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T \sum_i \log \frac{\mu_i(t^-)}{\lambda_i^\pi(\boldsymbol{y}(t^-))} dY_t^i - \int_0^T n \sum_{i,j=1}^{D} \Big(q_{ij}(\bar{\boldsymbol{z}}_n(t)) - \eta_{ij}(t) \Big) dt \\
&-\int_0^T \sum_{i,j} \log \frac{\eta_{ij}(t^-)}{q_{ij}(\bar{\boldsymbol{z}}_n(t^-))} dM_t^{ij} \Bigg\} \\
:=& e^{B(\bar{\boldsymbol{z}}_n, \bar{\xi}_n)},
\end{aligned} \tag{3.32}
$$

where Y_t^i is the counting process induced by $\bar{\boldsymbol{z}}_n(t)$ that will increase by one each time when a jump occurs in the $\boldsymbol{u}_i$ direction and M_t^{ij} is the counting process induced by $\bar{\xi}_n(t)$ that will increase by one each time when a jump occurs from state i to state j. The next lemma shows that the expectation of $B(\bar{\boldsymbol{z}}_n, \bar{\xi}_n)$ in the exponent becomes simple in the limit $n \to \infty$.

Lemma 3.13.

$$
\lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{Q}}_n} \int_0^T \lambda_i(\bar{\boldsymbol{z}}_n(t), \bar{\xi}_n(t)) - \mu_i(t) \frac{\lambda_i(\bar{z}_n(t), \bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))} dt = \sum_{m=0}^{J-1} \int_{t_m}^{t_{m+1}} \lambda_i^\pi(\boldsymbol{y}(t)) - \mu_i^m dt. \tag{3.33}
$$

$$
\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{\bar{\mathbb{Q}}_n} \int_0^T \sum_i \log \frac{\mu_i(t^-)}{\lambda_i^\pi(\boldsymbol{y}(t))} dY_t^i = \sum_{m=0}^{J-1} \int_{t_m}^{t_{m+1}} \sum_{i=1}^{S} \mu_i^m \log \frac{\mu_i^m}{\lambda_i^\pi(\boldsymbol{y}(t))} dt. \tag{3.34}
$$

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \mathbb{E}_{\bar{\mathbb{Q}}_n} \int_0^T \sum_{i,j} \log \frac{\eta_{ij}(t^-)}{q_{ij}(\bar{\boldsymbol{z}}_n(t^-))} dM_t^{ij} \\
=& \sum_{m=0}^{J-1} \sum_{k=0}^{K_m-1} \int_{t_{mk}}^{t_{m,k+1}} \sum_{i=1}^{D} n_\pi(t,i) \sum_{j=1}^{D} \eta_{ij}^{mk} \log \frac{\eta_{ij}^{mk}}{q_{ij}(\boldsymbol{y}(t))} dt.
\end{aligned} \tag{3.35}
$$

The proof of Lemma 3.13 is based on the ideas in proving Lemma 5.52 and Lemma 8.70 in [26].

Proof. Since $\mu_i(t)$ is a step function and constant in $[t_m, t_{m+1})$, to prove Equation (3.33), we just need to prove for each m

$$
\lim_{n\to\infty} \mathbb{E}_{\bar{\mathbb{Q}}_n} \int_{t_m}^{t_{m+1}} \lambda_i(\bar{\boldsymbol{z}}_n(t), \bar{\xi}_n(t)) - \mu_i^m \frac{\lambda_i(\bar{z}_n(t), \bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))} dt = \int_{t_m}^{t_{m+1}} \lambda_i^\pi(\boldsymbol{y}(t)) - \mu_i^m dt. \tag{3.36}
$$

Define

$$
N_\epsilon(\boldsymbol{y}) := \{ \boldsymbol{z} \in \mathbb{D}^d[0,T] : \rho_c^{(1)}(\boldsymbol{z}, \boldsymbol{y}) \le \epsilon \}.
$$

We have

$$
\begin{aligned}
&\mathbb{E}_{\bar{\mathbb{Q}}_n} \int_{t_m}^{t_{m+1}} \lambda_i(\bar{\boldsymbol{z}}_n(t), \bar{\xi}_n(t)) - \mu_i^m \frac{\lambda_i(\bar{\boldsymbol{z}}_n(t), \bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))} dt \\
=& \mathbb{E}_{\bar{\mathbb{Q}}_n} \chi_{\{\bar{\boldsymbol{z}}_n \in N_\epsilon(\boldsymbol{y})\}} \int_{t_m}^{t_{m+1}} \lambda_i(\bar{\boldsymbol{z}}_n(t), \bar{\xi}_n(t)) - \mu_i^m \frac{\lambda_i(\bar{\boldsymbol{z}}_n(t), \bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))} dt
\end{aligned}
$$

$$+\mathbb{E}_{\bar{\mathbb{Q}}_n}\chi_{\{\bar{\boldsymbol{z}}_n\notin N_\epsilon(\boldsymbol{y})\}}\int_{t_m}^{t_{m+1}}\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))-\mu_i^m\frac{\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))}dt. \tag{3.37}$$

By Lemma 3.12, the second term on the right-hand side of Equation (3.37) tends to zero as $n\to\infty$. Next let us estimate the first term.

By Assumption 2.2, we have

$$\begin{aligned}\left|\frac{\lambda_i(\boldsymbol{x}',j)}{\lambda_i^\pi(\boldsymbol{x})}-\frac{\lambda_i(\boldsymbol{z},j)}{\lambda_i^\pi(\boldsymbol{z})}\right|&\le\left|\frac{\lambda_i(\boldsymbol{x}',j)}{\lambda_i^\pi(\boldsymbol{x})}-\frac{\lambda_i(\boldsymbol{x}',j)}{\lambda_i^\pi(\boldsymbol{z})}\right|+\left|\frac{\lambda_i(\boldsymbol{x}',j)-\lambda_i(\boldsymbol{z},j)}{\lambda_i^\pi(\boldsymbol{z})}\right|\\&\le\Lambda^3L\|\boldsymbol{x}-\boldsymbol{z}\|+\Lambda L\|\boldsymbol{x}'-\boldsymbol{z}\|\end{aligned} \tag{3.38}$$

for any $\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}'\in\mathbb{R}^d$, $i\in\{1,\dots,S\}$ and $j\in\{1,2\dots,D\}$.

Now take an integer N and divide $[t_m,t_{m+1}]$ into N pieces. Define $\tau_l=t_m+l(t_{m+1}-t_m)/N$ for $l=0,\dots,N$. Since $\boldsymbol{y}$ is continuous in $[0,T]$, we can choose N large enough such that

$$\sup_{t\in[\tau_l,\tau_{l+1}]}\|\boldsymbol{y}(t)-\boldsymbol{y}(\tau_l)\|\le\epsilon\quad\text{for any } l\in\{0,\dots,N-1\}.$$

By Equations (2.1) and (3.38), we have

$$\begin{aligned}&\chi_{\{\bar{\boldsymbol{z}}_n\in N_\epsilon(\boldsymbol{y})\}}\int_{t_m}^{t_{m+1}}\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))-\mu_i^m\frac{\lambda_i(\bar{z}_n(t),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))}dt\\ \le&\chi_{\{\bar{\boldsymbol{z}}_n\in N_\epsilon(\boldsymbol{y})\}}\sum_{l=0}^{N-1}\int_{\tau_l}^{\tau_{l+1}}\lambda_i(\boldsymbol{y}(\tau_l),\bar{\xi}_n(t))-\mu_i^m\frac{\lambda_i(\boldsymbol{y}(\tau_l),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(\tau_l))}+C\epsilon dt,\end{aligned}$$

where $C=(2+2\Lambda+\Lambda^3)L$. So we have

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}_{\bar{\mathbb{Q}}_n}\int_{t_m}^{t_{m+1}}\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))-\mu_i^m\frac{\lambda_i(\bar{z}_n(t),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))}dt\\ \le&\sum_{l=0}^{N-1}\int_{\tau_l}^{\tau_{l+1}}\sum_{j=1}^{D}\lambda_i(\boldsymbol{y}(\tau_l),j)n_\pi(t,j)-\mu_i^m+C\epsilon dt\\ \le&\sum_{l=0}^{N-1}\int_{\tau_l}^{\tau_{l+1}}\sum_{j=1}^{D}\lambda_i(\boldsymbol{y}(t),j)n_\pi(t,j)-\mu_i^m+C_1\epsilon dt\\ =&\int_{t_m}^{t_{m+1}}\lambda_i^\pi(\boldsymbol{y}(t))-\mu_i^m dt+C_1(t_{m+1}-t_m)\epsilon\end{aligned}$$

by the ergodicity of the process $\bar{\xi}_n$, where $C_1=(3+2\Lambda+\Lambda^3)L$. Similarly, we can also obtain

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}_{\bar{\mathbb{Q}}_n}\int_{t_m}^{t_{m+1}}\lambda_i(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))-\mu_i^m\frac{\lambda_i(\bar{z}_n(t),\bar{\xi}_n(t))}{\lambda_i^\pi(\boldsymbol{y}(t))}dt\\ \ge&\int_{t_m}^{t_{m+1}}\lambda_i^\pi(\boldsymbol{y}(t))-\mu_i^m dt-C_1(t_{m+1}-t_m)\epsilon.\end{aligned}$$

So we finish the proof for Equation (3.33).

To prove Equation (3.34), we first assume that $\lambda_i(\boldsymbol{x})$ are constant functions. In $[t_m,t_{m+1})$, the number of jumps $\bar{\boldsymbol{z}}_n$ makes in each direction $\boldsymbol{u}_i/n$ are independent Poisson random variables with mean $n\mu_i^m(t_{m+1}-t_m)$. So

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\bar{\mathbb{Q}}_n}\int_{t_m}^{t_{m+1}}\sum_i \log\frac{\mu_i(t^-)}{\lambda_i^\pi(\boldsymbol{y}(t))}dY_t^i=\int_{t_m}^{t_{m+1}}\sum_{i=1}^S \mu_i^m\log\frac{\mu_i^m}{\lambda_i^\pi(\boldsymbol{y}(t))}dt. \tag{3.39}$$

For general λ_i, we can use the technique for proving Equation (3.33) by dividing the interval $[t_m,t_{m+1}]$ into small pieces and approximating Equation (3.34) by Riemann sums.

For Equation (3.35), again we first assume that $q_{ij}(\boldsymbol{x})$ are constant functions. In $[t_{mk},t_{m,k+1}]$, the number of jumps $\bar{\xi}_n$ makes in each direction $\boldsymbol{e}_{ij}$ are independent Poisson random variables with mean $n\cdot n_\pi(t_{mk},i)\eta_{ij}^{mk}(t_{m,k+1}-t_{mk})$. So

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\bar{\mathbb{Q}}_n}\int_{t_{mk}}^{t_{m,k+1}}\sum_{i,j}\log\frac{\eta_{ij}(t^-)}{q_{ij}(\bar{\boldsymbol{z}}_n(t^-))}dM_t^{ij}=\int_{t_{mk}}^{t_{m,k+1}}\sum_{i=1}^D n_\pi(t,i)\sum_{j=1}^D\eta_{ij}^{mk}\log\frac{\eta_{ij}^{mk}}{q_{ij}(\boldsymbol{y}(t))}dt. \tag{3.40}$$

For general q_{ij}, we consider separate cases $\{\bar{\boldsymbol{z}}_n\in N_\epsilon(\boldsymbol{y})\}$ and $\{\bar{\boldsymbol{z}}_n\notin N_\epsilon^c(\boldsymbol{y})\}$ as in Equation (3.37). Similar to proving Equation (3.33), we can get the limit (3.35). □

LEMMA 3.14. *For given $\boldsymbol{r}\in\mathbb{D}^d[0,T]$ and $\nu\in\mathbb{M}_L[0,T]$, assume that $\boldsymbol{r}$ is absolutely continuous and $n_\nu(t,\cdot)$ is continuous in t. Then, for arbitrarily small $\epsilon>0$, we have*

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_\epsilon(\nu))\geq-(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)).$$

Proof. By Equation (3.32) and Jensen's inequality, for any $\epsilon>0$,

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_\epsilon(\nu))\\
\geq&\mathbb{P}\left(\boldsymbol{z}_n\in N_{\epsilon/2}(\boldsymbol{y}),\nu_n\in N_{\epsilon/2}(\pi)\right)\\
=&\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[\frac{d\mathbb{Q}_n}{d\bar{\mathbb{Q}}_n}(\bar{\boldsymbol{z}}_n(t),\bar{\xi}_n(t))\chi_{\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\}}\right]\\
=&\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[e^{B(\bar{\boldsymbol{z}}_n,\bar{\xi}_n)}\chi_{\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\}}\right]\\
\geq&\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[\chi_{\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\}}\right]\exp\left\{\frac{\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[\chi_{\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\}}B(\bar{\boldsymbol{z}}_n,\bar{\xi}_n)\right]}{\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[\chi_{\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\}}\right]}\right\}.
\end{aligned} \tag{3.41}$$

By lemmas 3.11 and 3.12, we know that

$$\lim_{n\to\infty}\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[\chi_{\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\}}\right]=1. \tag{3.42}$$

Thus, according to Lemma 3.13 and Equations (3.41) and (3.42), we have

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_\epsilon(\nu))$$

$$
\begin{aligned}
&\geq \liminf_{n\to\infty}\frac{1}{n}\mathbb{E}_{\bar{\mathbb{Q}}_n}\left[\chi_{\left\{\bar{\boldsymbol{z}}_n\in N_{\epsilon/2}(\boldsymbol{y}),\bar{\nu}_n\in N_{\epsilon/2}(\pi)\right\}}B(\bar{\boldsymbol{z}}_n,\bar{\xi}_n)\right]\\
&=-\left(\sum_{m=0}^{J-1}\int_{t_m}^{t_{m+1}}\sum_{i=1}^{S}(\lambda_i^\pi(\boldsymbol{y}(t))-\mu_i^m)\,dt\right.\\
&\quad+\sum_{m=0}^{J-1}\int_{t_m}^{t_{m+1}}\sum_{i=1}^{S}\mu_i^m\log\frac{\mu_i^m}{\lambda_i^\pi(\boldsymbol{y}(t))}dt\\
&\quad\left.+\sum_{m=0}^{J-1}\sum_{k=0}^{K_m-1}\int_{t_{mk}}^{t_{m,k+1}}\sum_{i=1}^{D}n_\pi(t,i)\sum_{j=1}^{D}\left(\eta_{ij}^{mk}\log\frac{\eta_{ij}^{mk}}{q_{ij}(\boldsymbol{y}(t))}+q_{ij}(\boldsymbol{y}(t))-\eta_{ij}^{mk}\right)dt\right). \qquad (3.43)
\end{aligned}
$$

Combining Lemma 3.9, Lemma 3.10, and Equation (3.43), we finish the proof. ∎

In the final theorem, we remove the continuity assumption on $n_\nu(t,\cdot)$ to get the desired lower bound estimation.

THEOREM 3.2. *For given $\boldsymbol{r}\in\mathbb{D}^d[0,T]$ and $\nu\in\mathbb{M}_L[0,T]$, assume that $\boldsymbol{r}$ is absolutely continuous, we have*

$$
\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_\epsilon(\nu))\geq-(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)).
$$

Proof. We can construct a sequence of measures $\nu^{(k)}$ $(k\geq 1)$ such that for any k, $n_{\nu^{(k)}}$ is continuous in t and $\rho^{(2)}(\nu,\nu^{(k)})\to 0$. From Lemma A.8, $I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)$ is lower semicontinuous in ν. Thus, we can choose k_0 large enough such that for any $\delta>0$ and $\epsilon>0$,

$$
I_s(\boldsymbol{r},\nu^{(k_0)})+I_f(\boldsymbol{r},\nu^{(k_0))})\geq I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)-\delta
$$

and

$$
\rho^{(2)}(\nu,\nu^{(k_0)})<\epsilon/2.
$$

Thanks to Lemma 3.14, we have

$$
\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_\epsilon(\nu))\\
&\geq\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\boldsymbol{z}_n\in N_\epsilon(\boldsymbol{r}),\nu_n\in N_{\epsilon/2}(\nu^{(k_0)})\right)\\
&\geq-\left(I_s(\boldsymbol{r},\nu^{(k_0)})+I_f(\boldsymbol{r},\nu^{(k_0)})\right)\\
&\geq-(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu))-\delta.
\end{aligned}
$$

The proof is completed. ∎

3.3. Goodness of the rate functional. The rate functional $I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)$ is lower semicontinuous by Lemma A.8. The goodness of the rate functional is a direct consequence of Lemma 3.7.

4. Proof of Theorem 2.1

Now we prove Theorem 2.1 under the consideration $\boldsymbol{r}\in W=\overline{(\mathbb{R}^+)^d}$ instead of the whole space. The main clue of the proof is the same as the proof of Theorem 2.2 except some technicalities to understand the behavior of jumps near the boundary of W. We will only focus on the key parts, which is different from the proof of Theorem 2.2.

The difficulty in the proof of lower bound is that we cannot use the change of measure formula directly, since some of the jump rates may diminish on the boundary. Mainly following [27], we overcome this issue by carefully analyzing the boundary behavior of the dynamics.

Let a d-dimensional unit vector $\boldsymbol{v}:=(1,1,\ldots,1)/\sqrt{d}$ and define the shifting $\boldsymbol{r}_\delta(t)=\boldsymbol{r}(t)+\delta\boldsymbol{v}$ with $\delta>0$ a sufficiently small number. With a similar approach to proving [27, Lemma 5.1], we can show that

$$\limsup_{\delta\to 0^+}(I_s(\boldsymbol{r}_\delta,\nu)+I_f(\boldsymbol{r}_\delta,\nu))\le I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu). \tag{4.1}$$

Next, we will prove

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\boldsymbol{z}_n\in N_\delta(\boldsymbol{r}),\nu_n\in N_\delta(\nu))\ge -(I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)).$$

Denote by $V_a(\boldsymbol{r})$ the modulus of continuity of $\boldsymbol{r}$ with size a, and set $\eta(a)=\max\{V_a(\boldsymbol{r}),a\}$ so that $\eta^{-1}(a)\le a$. Now, fix δ and set $t_\delta=\eta^{-1}(\delta/3)$. Then, $t_\delta\le\delta/3$ and, for $t\le t_\delta$,

$$\sup_{0\le t\le t_\delta}\|\boldsymbol{r}(0)+t\cdot\boldsymbol{v}-\boldsymbol{r}(t)\|\le t_\delta\cdot\|\boldsymbol{v}\|+\eta(t_\delta)\le 2\delta/3.$$

Therefore, for $0<\alpha<1/6$,

$$\begin{aligned}\mathbb{P}(\boldsymbol{z}_n\in N_\delta(\boldsymbol{r}),\nu_n\in N_\delta(\nu))\ge\mathbb{P}\Big(&\|\boldsymbol{z}_n(t)-\boldsymbol{r}(0)-t\cdot\boldsymbol{v}\|\le\alpha\delta \text{ on } t\in[0,t_\delta],\\ &\boldsymbol{z}_n\in N_\delta(\boldsymbol{r};[t_\delta,T]);\nu_n\in N_\delta(\nu)\Big),\end{aligned}$$

where $N_\delta(\boldsymbol{r};[t_\delta,T])$ is the δ-neighborhood of $\boldsymbol{r}$ restricted on $t\in[t_\delta,T]$. Now, on this time interval,

$$\sup_{t_\delta\le t\le T}\|\boldsymbol{r}(t)-\boldsymbol{r}_{t_\delta}(t)\|\le\delta/3.$$

Moreover, $d(\boldsymbol{r}_{t_\delta}(t),\partial G)\ge t_\delta/\sqrt{d}$. Therefore, for any function $\boldsymbol{u}$ on $t\in[t_\delta,T]$, $\|\boldsymbol{u}-\boldsymbol{r}_{t_\delta}\|\le t_\delta/2\sqrt{d}$ implies that $\|\boldsymbol{u}-\boldsymbol{r}\|\le 5\delta/6$ and $d(\boldsymbol{r}_{t_\delta}(t),\partial G)\ge t_\delta/2\sqrt{d}$. Now define A_δ the $\alpha\delta$-neighborhood of $\boldsymbol{r}_0+t_\delta\boldsymbol{v}$, i.e. $A_\delta:=B_{\alpha\delta}(\boldsymbol{r}_0+t_\delta\boldsymbol{v})$, and let $\boldsymbol{r}^{\boldsymbol{y}}_{t_\delta}$ be the shift of $\boldsymbol{r}_{t_\delta}$ such that $\boldsymbol{r}^{\boldsymbol{y}}_{t_\delta}(t_\delta)=\boldsymbol{y}$. Then,

$$\begin{aligned}\mathbb{P}(\boldsymbol{z}_n\in N_\delta(\boldsymbol{r}),\nu_n\in N_\delta(\nu))\ge&\,\mathbb{P}\Big(\|\boldsymbol{z}_n(t)-\boldsymbol{r}(0)-t\cdot\boldsymbol{v}\|\le\alpha\delta \text{ on } t\in[0,t_\delta];\nu_n\in N_\delta(\nu;[0,t_\delta])\Big)\\ &\times\inf_{\boldsymbol{y}\in A_\delta}\mathbb{P}_{\boldsymbol{y}}\left(\boldsymbol{z}_n\in N_{\frac{t_\delta}{2\sqrt{d}}}(\boldsymbol{r}^{\boldsymbol{y}}_{t_\delta};[t_\delta,T]);\nu_n\in N_\delta(\nu;[t_\delta,T])\right).\end{aligned}$$

The first term satisfies a large deviation lower bound

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\|\boldsymbol{z}_n(t)-\boldsymbol{r}(0)-t\cdot\boldsymbol{v}\|\le\alpha\delta \text{ on } t\in[0,t_\delta];\nu_n\in N_\delta(\nu;[0,t_\delta])\Big)\\ &\ge -Ct_\delta\end{aligned} \tag{4.2}$$

by estimating the probability of a specific path $\boldsymbol{z}_n$ lying in the $\alpha\delta$-neighborhood of the curve $\boldsymbol{r}(0)+t\boldsymbol{v}$. Because the paths in $N_{\frac{t_\delta}{2\sqrt{d}}}(\boldsymbol{r}^{\boldsymbol{y}}_{t_\delta};[t_\delta,T])$ are bounded away from the boundary uniformly for $\boldsymbol{y}\in A_\delta$, by Theorem 3.2, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log\inf_{\boldsymbol{y}\in A_\delta}\mathbb{P}_{\boldsymbol{y}}\left(\boldsymbol{z}_n\in N_{\frac{t_\delta}{2\sqrt{d}}}(\boldsymbol{r}^{\boldsymbol{y}}_{t_\delta};[t_\delta,T]);\nu_n\in N_\epsilon(\nu;[t_\delta,T])\right)$$

$$\geq -\left(I_s^{[t_\delta,T]}(\boldsymbol{r}_{t_\delta},\nu)+I_f^{[t_\delta,T]}(\boldsymbol{r}_{t_\delta},\nu)\right)$$
$$\geq -\left(I_s(\boldsymbol{r}_{t_\delta},\nu)+I_f(\boldsymbol{r}_{t_\delta},\nu)\right), \tag{4.3}$$

where $I_s^{[t_\delta,T]}(\boldsymbol{r}_{t_\delta},\nu)$ and $I_f^{[t_\delta,T]}(\boldsymbol{r}_{t_\delta},\nu)$ are rate functionals defined on the integration interval $[t_\delta,T]$. According to Equations (4.1), (4.2), and (4.3), we have proved the lower bound.

Next, let us consider the upper bound. At first, we note that, since the rates $\lambda_i(\boldsymbol{z},j)$ satisfies the linear growth condition

$$\lambda_i(\boldsymbol{z},j)\leq C(1+\|\boldsymbol{z}\|),$$

it is easy to show that

$$\lim_{K\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\sup_{0\leq t\leq T}\|\boldsymbol{z}_n(t)\|>K)=-\infty$$

by simple moment estimates and Doob's martingale inequality. Consequently, it suffices to prove the large deviation estimates for bounded sets, and we can assume $\lambda_i(\boldsymbol{z},j)$ are bounded.

We only need to recheck Lemma 3.8 and Lemma A.9, since the other lemmas in upper bound estimates can be verified easily under the assumption that $\lambda_i(\boldsymbol{z},j)$ are bounded. Thanks to Corollary 4.2 and Lemma 4.6 in [27], we can obtain that Lemma 3.8 and Lemma A.9 are also correct under Assumption 2.1. Thus, the upper bound is also established.

The goodness of the rate functional trivially holds under Assumption 2.1. So we complete the proof of Theorem 2.1.

Appendix A. In this appendix, we will supplement the proof of the main lemmas.

LEMMA A.1. *Let $\{f_\alpha\}$ be a collection of lower semicontinuous functions on a metric space. Then the function f defined by $f(x)=\sup_\alpha f_\alpha(x)$ is lower semicontinuous.*

LEMMA A.2. *Let $\{f_\alpha\}$ be a collection of convex functions on a metric space. Then the function f define by $f(x)=\sup_\alpha f_\alpha(x)$ is convex.*

LEMMA A.3. *Let $K(x,y)$ be a real-valued function, continuous in (x,y) on $\mathbb{R}^d\times\mathbb{R}^D$, convex in x for each y, and concave in y for each x. Let two non-empty closed convex sets U and V be given, at least one of which is bounded. Then*

$$\inf_{x\in U}\sup_{y\in V}K(x,y)=\sup_{y\in V}\inf_{x\in U}K(x,y).$$

The proof of Lemma A.3 may be found in [24, Corollary 37.3.2].

A.1. Part 1. Proof of lemmas related to the upper bound estimate.

LEMMA A.4. *Let $\boldsymbol{z}(t)\in\mathbb{R}^d$ be any measurable process for $t\in[0,T]$. Suppose there exist numbers a and δ such that, for each $\boldsymbol{p}\in\mathbb{R}^d$ with $\|\boldsymbol{p}\|=1$,*

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\langle\boldsymbol{z}(t),\boldsymbol{p}\rangle\geq a\right)\leq\delta.$$

Then

$$\mathbb{P}\left(\sup_{0\leq t\leq T}\|\boldsymbol{z}(t)\|\geq a\sqrt{d}\right)\leq 2d\delta.$$

Proof. It is not difficult to find that

$$\Big\{\sup_{0\le t\le T}\|\boldsymbol{z}(t)\|\ge a\sqrt{d}\Big\}\subset\bigcup_{i=1}^{2d}\Big\{\sup_{0\le t\le T}\langle\boldsymbol{z}(t),\boldsymbol{p}_i\rangle\ge a\Big\},$$

where $\boldsymbol{p}_i:=\boldsymbol{e}_i$, $\boldsymbol{p}_{i+d}:=-\boldsymbol{e}_i$ for $i=1,\dots,d$, and $\boldsymbol{e}_i$ are chosen as the canonical orthonormal basis in Euclidean space $\mathbb{R}^d$. ∎

In later texts, we will abuse notation by denoting $\xi_n(t)=\boldsymbol{e}_i\in\mathbb{R}^D$ when $\xi_n(t)=i\in\mathbb{Z}_D$. This will not bring confusion since $\xi_n(t)$ is considered as a multidimensional vector only when we take the inner product with other vectors.

LEMMA A.5. *There exists a function $K:\mathbb{R}^+\to\mathbb{R}^+$ with*

$$\lim_{a\to\infty}K(a)/a=+\infty,$$

such that

$$\mathbb{P}\left(\sup_{0\le t\le T}\|\boldsymbol{z}_n(t)-\boldsymbol{z}_n(0)\|\ge a\right)\le 2d\exp\left(-nTK\left(\frac{a}{T}\right)\right). \tag{A.1}$$

Proof. The inequality (A.1) holds trivially whenever $K(a/T)=0$. It suffices to prove the lemma when a is large. For $\boldsymbol{p}\in\mathbb{R}^d$, $\boldsymbol{\sigma}\in\mathbb{R}^D$, and any $\rho>0$, with the form of infinitesimal generator $\mathcal{L}_n$ (1.2), we define a mean one exponential martingale

$$\begin{aligned}M_t^{\boldsymbol{\sigma}}=\exp\Bigg(&\langle\boldsymbol{z}_n(t)-\boldsymbol{z}_n(0),\rho\boldsymbol{p}\rangle-n\int_0^t\sum_{i=1}^S\lambda_i(\boldsymbol{z}_n(s),\xi_n(s))(e^{\langle\rho\boldsymbol{p},\boldsymbol{u}_i/n\rangle}-1)ds\\&+\langle\xi_n(t)-\xi_n(0),\boldsymbol{\sigma}\rangle-n\int_0^t\sum_{i=1}^D\chi_{\{\xi_n(s)=i\}}\sum_{j=1}^D q_{ij}(\boldsymbol{z}_n(s))(e^{\langle\boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle}-1)ds\Bigg).\end{aligned}$$

Define $U=\max_{1\le i\le S}\|\boldsymbol{u}_i\|$. Fix $\|\boldsymbol{p}\|=1$, and we have

$$n\int_0^t\sum_{i=1}^S\lambda_i(\boldsymbol{z}_n(s),\xi_n(s))(e^{\langle\rho\boldsymbol{p},\boldsymbol{u}_i/n\rangle}-1)ds\le ntS\Lambda e^{U\rho/n}=:R(t,\rho)$$

by Assumption 2.2. Hence, we obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T}\langle\boldsymbol{z}_n(t)-\boldsymbol{z}_n(0),\boldsymbol{p}\rangle\ge a\right)\\=&\mathbb{P}\left(\sup_{0\le t\le T}\exp(\rho\langle\boldsymbol{z}_n(t)-\boldsymbol{z}_n(0),\boldsymbol{p}\rangle)\ge\exp(\rho a)\right)\\\le&\mathbb{P}\left(\sup_{0\le t\le T}M_t^{\boldsymbol{\sigma}=0}\ge\exp\Big(\rho a-R(T,\rho)\Big)\right)\\\le&\exp\left(nT\left[S\Lambda e^{U\rho/n}-\frac{\rho}{n}\frac{a}{T}\right]\right),\end{aligned}$$

where the inequality follows from Doob's martingale inequality. Take

$$\rho=\frac{n}{U}\log\frac{a}{TS\Lambda U}>0.$$

Then it is not difficult to show that, if we set

$$\tilde{K}(a)=\frac{a}{U}\left(\log\frac{a}{S\Lambda U}-1\right)$$

for a large and $K(a)=0$ otherwise, then

$$\mathbb{P}\left(\sup_{0\le t\le T}\langle \boldsymbol{z}_n(t)-\boldsymbol{z}_n(0),\boldsymbol{p}\rangle\ge a\right)\le \exp\left(-nT\tilde{K}\left(\frac{a}{T}\right)\right).$$

Define $K(a)=\tilde{K}(a/\sqrt{d})$, and we get the desired estimate by applying Lemma A.4. ∎

Corollary A.1. *There exist positive constants c_1 and c_2 independent of t and τ, such that, for any $t,\tau\in[0,T]$ with $0\le t+\tau\le T$,*

$$\mathbb{P}\Big(\sup_{t\le s\le t+\tau}\|\boldsymbol{z}_n(s)-\boldsymbol{z}_n(t)\|\ge a\Big)\le 2d\exp\left(-nac_1\log\left(\frac{ac_2}{\tau}\right)\right).$$

Proof. **(Proof of Lemma 3.1.)** Consider a typical interval $[t_j^n,t_{j+1}^n]$. Since $\boldsymbol{z}_n(t)$ and $\tilde{\boldsymbol{z}}_n(t)$ agree at the endpoints of this interval, it is obvious that

$$\|\tilde{\boldsymbol{z}}_n(t_j^n)-\tilde{\boldsymbol{z}}_n(t_{j+1}^n)\|>\frac{\delta}{2}\quad\text{implies}\quad\|\boldsymbol{z}_n(t_{j+1}^n)-\boldsymbol{z}_n(t_j^n)\|>\frac{\delta}{2}.$$

On the other hand, we have

$$\|\boldsymbol{z}_n(t)-\boldsymbol{z}_n(t_j^n)\|\ge\|\boldsymbol{z}_n(t)-\tilde{\boldsymbol{z}}_n(t)\|-\|\tilde{\boldsymbol{z}}_n(t_{j+1}^n)-\tilde{\boldsymbol{z}}_n(t_j^n)\|$$

since $\tilde{\boldsymbol{z}}_n$ is piecewise linear and $\tilde{\boldsymbol{z}}_n(t_j^n)=\boldsymbol{z}_n(t_j^n)$. Therefore, if $\|\boldsymbol{z}_n(t)-\tilde{\boldsymbol{z}}_n(t)\|>\delta$ for some t in the jth interval, we must have

$$\sup_{t_j^n\le t\le t_{j+1}^n}\|\boldsymbol{z}_n(t)-\boldsymbol{z}_n(t_j^n)\|\ge\delta/2.$$

Applying Corollary A.1 with $a=\delta/2$ and $\tau=T/n$, we obtain

$$\mathbb{P}\left(\sup_{t_j^n\le t\le t_{j+1}^n}\|\boldsymbol{z}_n(t)-\boldsymbol{z}_n(t_j^n)\|\ge\delta/2\right)\le 2d\exp\left(-n\frac{\delta c_1}{2}\log\left(\frac{n\delta c_3}{2}\right)\right),$$

where $c_3=c_2/T$. Thus,

$$\begin{aligned}\mathbb{P}(\rho^{(1)}(\boldsymbol{z}_n,\tilde{\boldsymbol{z}}_n)>\delta)&\le\sum_{j=0}^{n-1}\mathbb{P}\left(\sup_{t_j^n\le t\le t_{j+1}^n}\|\boldsymbol{z}_n(t)-\tilde{\boldsymbol{z}}_n(t)\|>\delta\right)\\&\le\sum_{j=0}^{n-1}\mathbb{P}\left(\sup_{t_j^n\le t\le t_{j+1}^n}\|\boldsymbol{z}_n(t)-\boldsymbol{z}_n(t_j^n)\|>\delta/2\right)\\&\le n\cdot 2d\exp\left(-n\frac{\delta c_1}{2}\log\left(\frac{n\delta c_3}{2}\right)\right).\end{aligned}$$

The result follows since c_1 and c_3 are positive constants. ∎

Lemma A.6. *For any given bounded sets $A_1\in\mathbb{R}^d$ and $A_2\in\mathbb{R}^D$, we have that*

$$\limsup_{n\to\infty}\mathbb{E}_{\boldsymbol{x},m}\exp\left\{n\left\langle\tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle-n\int_0^{T/n}H_s^\delta(\boldsymbol{x},\boldsymbol{p},n_{\nu_n}(t,\cdot))\,dt\right.$$

$$+\left\langle \xi_n\left(\frac{T}{n}\right)-\xi_n(0),\boldsymbol{\sigma}\right\rangle+n\int_0^{T/n}S^\delta(\boldsymbol{x},n_{\nu_n}(t,\cdot),\boldsymbol{\sigma})dt\Bigg\}\le 1$$

holds uniformly in $\boldsymbol{x}\in\mathbb{R}^d$, $m\in\{1,2,\dots,D\}$, $\boldsymbol{p}\in A_1$, *and* $\boldsymbol{\sigma}\in A_2$, *where* $\mathbb{E}_{\boldsymbol{x},m}$ *means the expectation with respect to the paths of* $(\boldsymbol{z}_n,\xi_n)$ *starting from* $(\boldsymbol{x},m)$ *at* $t=0$.

Proof. For any $\boldsymbol{p}\in A_1$ and $\boldsymbol{\sigma}\in A_2$, define the mean one exponential martingale

$$M_t=\exp\Bigg(n\left[\langle \boldsymbol{z}_n(t)-\boldsymbol{z}_n(0),\boldsymbol{p}\rangle-\int_0^t\sum_{i=1}^S\lambda_i(\boldsymbol{z}_n(s),\xi_n(s))(e^{\langle \boldsymbol{p},\boldsymbol{u}_i\rangle}-1)ds\right]$$
$$+\langle \xi_n(t)-\xi_n(0),\boldsymbol{\sigma}\rangle-n\int_0^t\sum_{i=1}^D\chi_{\{\xi_n(s)=i\}}\sum_{j=1}^D q_{ij}(\boldsymbol{z}_n(s))(e^{\langle\boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle}-1)ds\Bigg).$$

Since $\tilde{\boldsymbol{z}}_n(t_j^n)=\boldsymbol{z}_n(t_j^n)$, for any $\boldsymbol{p}\in A_1$, we have

$$1=\mathbb{E}_{\boldsymbol{x},m}\exp\Bigg\{n\Bigg[\left\langle\tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle$$
$$-\int_0^{T/n}\sum_{i=1}^S\sum_{j=1}^D\lambda_i(\boldsymbol{z}_n(s),j)(e^{\langle\boldsymbol{p},\boldsymbol{u}_i\rangle}-1)\nu_n(ds,j)\Bigg]$$
$$+\left\langle\xi_n\left(\frac{T}{n}\right)-\xi_n(0),\boldsymbol{\sigma}\right\rangle-n\int_0^{T/n}\sum_{i,j=1}^D q_{ij}(\boldsymbol{z}_n(s))(e^{\langle\boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle}-1)\nu_n(ds,i)\Bigg\}.$$

By definition, the term

$$\sum_{i=1}^S\sum_{j=1}^D\lambda_i(\boldsymbol{z}_n(s),j)(e^{\langle\boldsymbol{p},\boldsymbol{u}_i\rangle}-1)\nu_n(ds,j)$$

can be written as $H_s(\boldsymbol{z}_n(s),\boldsymbol{p},n_{\nu_n}(s,\cdot))ds$ and

$$-\sum_{i=1}^D\sum_{j=1}^D q_{ij}(\boldsymbol{z}_n(s))(e^{\langle\boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle}-1)\nu_n(ds,i)$$

can be written as $S(\boldsymbol{z}_n(s),n_{\nu_n}(s,\cdot),\boldsymbol{\sigma})ds$.

Let

$$S_\delta=\left\{\omega:\sup_{0\le t\le T/n}\|\boldsymbol{z}_n(t)-\boldsymbol{x}\|<\frac{\delta}{2}\right\},$$

and we have

$$1\geqslant\mathbb{E}_{\boldsymbol{x},m}\chi_{S_\delta}\exp\Bigg\{\Bigg(n\left\langle\tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle-n\int_0^{T/n}H_s^\delta(\boldsymbol{x},\boldsymbol{p},n_{\nu_n}(t,\cdot))dt\Bigg)$$
$$+\left\langle\xi_n\left(\frac{T}{n}\right)-\xi_n(0),\boldsymbol{\sigma}\right\rangle+n\int_0^{T/n}S^\delta(\boldsymbol{x},n_{\nu_n}(t,\cdot),\boldsymbol{\sigma})dt\Bigg\}$$

$$
\begin{aligned}
&=\mathbb{E}_{\boldsymbol{x},m}\exp\left\{\left(n\left\langle \tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle - n\int_0^{T/n} H_s^\delta(\boldsymbol{x},\boldsymbol{p},n_{\nu_n}(t,\cdot))\,dt\right)\right.\\
&\quad\left.+\left\langle \xi_n\left(\frac{T}{n}\right)-\xi_n(0),\boldsymbol{\sigma}\right\rangle + n\int_0^{T/n} S^\delta(\boldsymbol{x},n_{\nu_n}(t,\cdot),\boldsymbol{\sigma})dt\right\}\\
&\quad-\mathbb{E}_{\boldsymbol{x},m}\chi_{S_\delta^c}\exp\left\{\left(n\left\langle \tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle - n\int_0^{T/n} H_s^\delta(\boldsymbol{x},\boldsymbol{p},n_{\nu_n}(t,\cdot))\,dt\right)\right.\\
&\quad\left.+\left\langle \xi_n\left(\frac{T}{n}\right)-\xi_n(0),\boldsymbol{\sigma}\right\rangle + n\int_0^{T/n} S^\delta(\boldsymbol{x},n_{\nu_n}(t,\cdot),\boldsymbol{\sigma})dt\right\}. \qquad (A.2)
\end{aligned}
$$

Since A_1 and A_2 are bounded sets, there exist B_1 and B_2 such that $\|\boldsymbol{p}\|\le B_1$ and $\|\boldsymbol{\sigma}\|\le B_2$. From Assumption 2.2 and the boundedness of $\boldsymbol{p}$ and $\boldsymbol{\sigma}$, we have

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{x},m}\chi_{S_\delta^c}\exp\left\{\left(n\left\langle \tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle - n\int_0^{T/n} H_s^\delta(\boldsymbol{x},\boldsymbol{p},n_{\nu_n}(t,\cdot))\,dt\right)\right.\\
&\quad\left.+\left\langle \xi_n\left(\frac{T}{n}\right)-\xi_n(0),\boldsymbol{\sigma}\right\rangle + n\int_0^{T/n} S^\delta(\boldsymbol{z}_n(t),n_{\nu_n}(t,\cdot),\boldsymbol{\sigma})dt\right\}\\
&\le\mathbb{E}_{\boldsymbol{x},m}\left(\chi_{S_\delta^c}\exp\left(n\left\langle \tilde{\boldsymbol{z}}_n\left(\frac{T}{n}\right)-\tilde{\boldsymbol{z}}_n(0),\boldsymbol{p}\right\rangle+3K\right)\right)\\
&\leqslant\sum_{k=1}^{\infty}\exp\left(n(k+1)\frac{\delta}{2}|\boldsymbol{p}|+3K\right)\times\mathbb{P}\left(\frac{k\delta}{2}\leqslant\sup_{0\leqslant t\leqslant T/n}|\boldsymbol{z}_n(t)-\boldsymbol{x}|\leqslant\frac{(k+1)\delta}{2}\right)\\
&\leqslant\sum_{k=1}^{\infty}2d\exp\left(n\left((k+1)\frac{\delta}{2}B_1-\frac{k\delta c_1}{2}\log\left(\frac{k\delta c_2 n}{2T}\right)\right)\right)\times e^{3K}\to 0 \qquad (A.3)
\end{aligned}
$$

as n goes to infinity for all $\boldsymbol{x}\in\mathbb{R}^d$ with $\|\boldsymbol{p}\|\le B_1$ and $\|\boldsymbol{\sigma}\|\le B_2$, where K is a uniform bound depending on the bounds of $S^\delta(\cdot,\cdot,\cdot)$ and $H_s^\delta(\cdot,\cdot,\cdot)$ in the whole space, B_1, B_2, and T. Combining Equations (A.3) and (A.2), we complete the proof. ∎

Corollary A.2. *For any fixed step functions $\boldsymbol{\theta}(t)\in\mathbb{R}^d$ and $\boldsymbol{\alpha}(t)\in\mathbb{R}^D$, there exist constants $C>0$ and n_0 such that*

$$\mathbb{E}\exp\{nJ_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha})\}\le C$$

for all $n>n_0$, where J_n is defined in Equation (3.9).

Proof. By definition,

$$
\begin{aligned}
&\exp\{nJ_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha})\}\\
&=\exp\left\{\sum_{j=0}^{n-1}\left(n\langle\tilde{\boldsymbol{z}}_n(t_{j+1}^n)-\tilde{\boldsymbol{z}}_n(t_j^n),\boldsymbol{\theta}(t_j^n)\rangle - n\int_{t_j^n}^{t_j^{n+1}} H_s^\delta\left(\tilde{\boldsymbol{z}}_n(t_j^n),\boldsymbol{\theta}(t_j^n),n_{\nu_n}(t,\cdot)\right)dt\right.\right.\\
&\quad\left.\left.+\,n\int_{t_j^n}^{t_j^{n+1}} S^\delta(\tilde{\boldsymbol{z}}_n(t_j^n),n_{\nu_n}(t,\cdot),\boldsymbol{\alpha}(t_j^n))dt\right)\right\}\\
&=\exp\left\{\sum_{j=0}^{n-1}\left(n\langle\tilde{\boldsymbol{z}}_n(t_{j+1}^n)-\tilde{\boldsymbol{z}}_n(t_j^n),\boldsymbol{\theta}(t_j^n)\rangle - n\int_{t_j^n}^{t_j^{n+1}} H_s^\delta\left(\tilde{\boldsymbol{z}}_n(t_j^n),\boldsymbol{\theta}(t_j^n),n_{\nu_n}(t,\cdot)\right)dt\right.\right.
\end{aligned}
$$

$$+\left\langle \xi_n(t_{j+1}^n)-\xi_n(t_j^n),\boldsymbol{\alpha}(t_j^n)\right\rangle + n\int_{t_j^n}^{t_j^{n+1}} S^\delta(\tilde{\boldsymbol{z}}_n(t_j^n),n_{\nu_n}(t,\cdot),\boldsymbol{\alpha}(t_j^n))dt\Bigg)$$
$$-\sum_{j=0}^{n-1}\left\langle \xi_n(t_{j+1}^n)-\xi_n(t_j^n),\boldsymbol{\alpha}(t_j^n)\right\rangle\Bigg\}. \tag{A.4}$$

Now, $\boldsymbol{\alpha}$ is a step function. Let us first consider $\alpha(t)=\alpha_0$ on the interval $t\in[0,\tau]$. We have

$$\sum_{j=0}^{n-1}\chi_{\{t_{j+1}^n\le\tau\}}\left\langle \xi_n(t_{j+1}^n)-\xi_n(t_j^n),\boldsymbol{\alpha}_0\right\rangle = \left\langle \xi_n\left(\frac{\lfloor n\tau\rfloor}{n}\right)-\xi_n(0),\boldsymbol{\alpha}_0\right\rangle,$$

where $\lfloor a\rfloor$ is the largest integer smaller than a. Since ξ_n and $\boldsymbol{\alpha}$ are bounded in $[0,T]$, $\left|\left\langle \xi_n\left(\frac{\lfloor n\tau\rfloor}{n}\right)-\xi_n(0),\boldsymbol{\alpha}_0\right\rangle\right|$ is uniformly bounded. Repeating this argument on the finite number of intervals on which $\boldsymbol{\alpha}$ are constants, we have that

$$\left|\sum_{j=0}^{n-1}\left\langle \xi_n(t_{j+1}^n)-\xi_n(t_j^n),\boldsymbol{\alpha}(t_j^n)\right\rangle\right|$$

is bounded. Thus by Equation (A.4), Lemma A.6 and the Markov property of $(\boldsymbol{z}_n,\xi_n)$,

$$\limsup_{n\to\infty}\mathbb{E}\exp\{nJ_n(\tilde{\boldsymbol{z}}_n,\boldsymbol{\theta},\nu_n,\boldsymbol{\alpha})\}\le C$$

where C is a positive constant. ∎

LEMMA A.7 (Uniformly absolute continuity). *Given $\boldsymbol{r}\in\mathbb{D}^d[0,T]$ and $\nu\in\mathbb{M}_L[0,T]$. Let $I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)\le K$ and fix some $\epsilon>0$. Then there is a $\delta>0$, independent of $\boldsymbol{r}$, such that, for any collection of non-overlapping intervals in $[0,T]$ with total length δ*

$$\Big\{[t_j,s_j], j=1,\dots,J\Big\}\quad\text{with}\quad \sum_{j=1}^{J}(s_j-t_j)=\delta,$$

we have

$$\sum_{j=1}^{J}\|\boldsymbol{r}(s_j)-\boldsymbol{r}(t_j)\|<\epsilon.$$

We can also find a constant B depending only on ϵ and K so that

$$\int_0^T \chi_{\{\|\dot{\boldsymbol{r}}(t)\|\ge B\}}dt\le\epsilon.$$

Proof. For any collection of non-overlapping intervals $\big\{[t_j,s_j]\big\}_j$, define the function $k(t)$ to be equal to one if t is in some interval $[t_j,s_j]$ and zero otherwise. Since $I_s(\boldsymbol{r},\nu)+I_f(\boldsymbol{r},\nu)\le K$, $\boldsymbol{r}$ is absolutely continuous and $I_s(\boldsymbol{r},\nu)\le K$. For any $a>0$,

$$\sum_{j=1}^{J}\|\boldsymbol{r}(s_j)-\boldsymbol{r}(t_j)\|\le\int_0^T\|\dot{\boldsymbol{r}}(t)\|k(t)dt$$

$$\leq \int_0^T a\cdot\chi_{\{\|\dot{\boldsymbol r}(t)\|\leq a\}}k(t)dt$$
$$+\int_0^T \frac{L_s(\boldsymbol r(t),\dot{\boldsymbol r}(t),n_\nu(t,\cdot))}{L_s(\boldsymbol r(t),\dot{\boldsymbol r}(t),n_\nu(t,\cdot))/|\dot{\boldsymbol r}(t)|}\chi_{\{\|\dot{\boldsymbol r}(t)\|>a\}}k(t)dt$$
$$\leq a\cdot\delta+\frac{K}{f(a)},$$

where

$$f(a):=\inf_{\boldsymbol z,\boldsymbol\beta,\boldsymbol w\in\Delta_D}\left\{\frac{L_s(\boldsymbol z,\boldsymbol\beta,\boldsymbol w)}{\|\boldsymbol\beta\|}:\|\boldsymbol\beta\|\geq a\right\}.$$

Recalling the definition of $L_s(\boldsymbol z,\boldsymbol\beta,\boldsymbol w)$ in Equation (2.6), we define $U:=\max_i\|\boldsymbol u_i\|$. For any $\boldsymbol w\in\Delta_D$, if we take $\boldsymbol p=\boldsymbol\beta\log\|\boldsymbol\beta\|/(U\|\boldsymbol\beta\|)$ in Equation (2.6), we obtain

$$L_s(\boldsymbol z,\boldsymbol\beta,\boldsymbol w)\geq\|\boldsymbol\beta\|\log\|\boldsymbol\beta\|/U-\|\boldsymbol\beta\|S\Lambda.$$

This means that $f(a)\to\infty$ as $a\to\infty$. The choice $a=1/\sqrt{\delta}$ and taking δ sufficiently small establishes the uniformly absolute continuity.

Now we turn to the second statement. Since

$$\int_0^T\chi_{\{\|\dot{\boldsymbol r}(t)\|\geq B\}}dt\leq\frac{1}{B}\int_0^T\|\dot{\boldsymbol r}(t)\|\chi_{\{\|\dot{\boldsymbol r}(t)\|\geq B\}}dt$$
$$\leq\frac{1}{B}\int_0^T\frac{L_s(\boldsymbol r(t),\dot{\boldsymbol r}(t),n_\nu(t,\cdot))}{L_s(\boldsymbol r(t),\dot{\boldsymbol r}(t),n_\nu(t,\cdot))/\|\dot{\boldsymbol r}(t)\|}\chi_{\{\|\dot{\boldsymbol r}(t)\|\geq B\}}dt$$
$$\leq\frac{1}{B}\frac{I_s(\boldsymbol r,\nu)}{f(B)},$$

we complete the proof by choosing a sufficiently large B. □

Lemma A.8. *The rate functionals are lower semicontinuous, i.e., if $(\boldsymbol r_n,\nu_n)\to(\boldsymbol r,\nu)$ as $n\to\infty$, then*

$$\liminf_{n\to\infty}I_s(\boldsymbol r_n,\nu_n)\geq I_s(\boldsymbol r,\nu),\quad \liminf_{n\to\infty}I_f(\boldsymbol r_n,\nu_n)\geq I_f(\boldsymbol r,\nu), \tag{A.5}$$

$$\liminf_{n\to\infty}I_s^\delta(\boldsymbol r_n,\nu_n)\geq I_s^\delta(\boldsymbol r,\nu),\quad \liminf_{n\to\infty}I_f^\delta(\boldsymbol r_n,\nu_n)\geq I_f^\delta(\boldsymbol r,\nu) \tag{A.6}$$

and

$$\liminf_{n\to\infty}I_s^\delta(\boldsymbol r_n,\nu_n,\boldsymbol\theta)\geq I_s^\delta(\boldsymbol r,\nu,\boldsymbol\theta),\quad \liminf_{n\to\infty}I_f(\boldsymbol r_n,\nu_n,\boldsymbol\alpha)\geq I_f^\delta(\boldsymbol r,\nu,\boldsymbol\alpha) \tag{A.7}$$

for any fix step functions $\boldsymbol\theta(t)\in R^d$ and $\boldsymbol\alpha(t)\in R^D$.

Proof. We only need to consider the sequences of $\boldsymbol r_n$ which are absolutely continuous since it will be trivial otherwise. Let $(\boldsymbol r_n,\nu_n)\to(\boldsymbol r,\nu)$ as $n\to\infty$. We may assume that $I_s(\boldsymbol r_n,\nu_n)+I_f(\boldsymbol r_n,\nu_n)$ is bounded, say by a constant K. By Lemma A.7, we know that $\boldsymbol r$ is also absolutely continuous.

Since $\boldsymbol r(t)$ is absolutely continuous in $[0,T]$, given δ, we can partition the interval $[0,T]$ into J intervals $0=t_1\leq t_2\leq\cdots\leq t_{J+1}=T$ each of length Δ such that

$$\max_j\sup_{t_j\leq t\leq t_{j+1}}\|\boldsymbol r_n(t)-\boldsymbol r_n(t_j)\|<\delta.$$

Denote $F_n(t,i)=\nu_n([0,t],i)$ and $F(t,i)=\nu([0,t],i)$. Recalling the definition of $L_s^\delta(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$ in Equation (3.5), we have that $L_s^\delta(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}))$ is lower semicontinuous in δ, $\boldsymbol{z}$, $\boldsymbol{\beta}$, and $\boldsymbol{w}$ and convex in $\boldsymbol{\beta}$ and $\boldsymbol{w}$ by lemmas A.1 and A.2. Thus, for any $\epsilon>0$ and small enough Δ, we have

$$\begin{aligned}
&\int_0^T L_s(\boldsymbol{r}_n(t),\dot{\boldsymbol{r}}_n(t),n_{\nu_n}(t,\cdot))dt\\
\geq&\sum_{j=1}^J\int_{t_j}^{t_{j+1}} L_s^\delta(\boldsymbol{r}_n(t_j),\dot{\boldsymbol{r}}_n(t),n_{\nu_n}(t,\cdot))dt\\
\geq&\sum_{j=1}^J \Delta\cdot L_s^\delta\left(\boldsymbol{r}_n(t_j),\frac{\int_{t_j}^{t_{j+1}}\dot{\boldsymbol{r}}_n(t)dt}{\Delta},\frac{\int_{t_j}^{t_{j+1}}n_{\nu_n}(t,\cdot)dt}{\Delta}\right)dt\\
=&\sum_{j=1}^J \Delta\cdot L_s^\delta\left(\boldsymbol{r}_n(t_j),\frac{\boldsymbol{r}_n(t_{j+1})-\boldsymbol{r}_n(t_j)}{\Delta},\frac{F_n(t_{j+1},\cdot)-F_n(t_j,\cdot)}{\Delta}\right).
\end{aligned}\tag{A.8}$$

Define the functions $\boldsymbol{r}_J$, F_J as

$$\boldsymbol{r}_J(t)=\boldsymbol{r}(t_j),\quad F_J(t,\cdot)=F(t_j,\cdot)\quad \text{for } t_j\leq t<t_{j+1}, j=1,\dots,J$$

and let

$$\boldsymbol{r}^J(t):=\boldsymbol{r}_J(t+\Delta),\quad F^J(t,\cdot):=F_J(t+\Delta,\cdot)\quad \text{for } 0\leq t<T-\Delta.$$

By Equation (A.8), we have

$$\begin{aligned}
&\liminf_{n\to\infty}\int_0^T L_s(\boldsymbol{r}_n(t),\dot{\boldsymbol{r}}_n(t),n_{\nu_n}(t,\cdot))dt\\
\geq&\sum_{j=1}^J\int_{t_j}^{t_{j+1}}\liminf_{n\to\infty} L_s^\delta\left(\boldsymbol{r}_n(t_j),\frac{\boldsymbol{r}_n(t_{j+1})-\boldsymbol{r}_n(t_j)}{\Delta},\frac{F_n(t_{j+1},\cdot)-F_n(t_j,\cdot)}{\Delta}\right)dt\\
\geq&\sum_{j=1}^{J-1}\int_{t_j}^{t_{j+1}} L_s^\delta\left(\boldsymbol{r}_J(t),\frac{\boldsymbol{r}^J(t)-\boldsymbol{r}_J(t)}{\Delta},\frac{F^J(t,\cdot)-F_J(t,\cdot)}{\Delta}\right)dt\\
=&\int_0^{T-\Delta} L_s^\delta\left(\boldsymbol{r}_J(t),\frac{\boldsymbol{r}^J(t)-\boldsymbol{r}_J(t)}{\Delta},\frac{F^J(t,\cdot)-F_J(t,\cdot)}{\Delta}\right)dt.
\end{aligned}$$

Now we use the nested partitions $J_k=2^k$, so that $\Delta_k=T/2^k$ and a corresponding sequence δ_k converges to zero. By Fatou's Lemma,

$$\begin{aligned}
&\liminf_{k\to\infty}\int_0^{T-\Delta_k} L_s^{\delta_k}\left(\boldsymbol{r}_{J_k}(t),\frac{\boldsymbol{r}^{J_k}(t)-\boldsymbol{r}_{J_k}(t,\cdot)}{\Delta},\frac{F^{J_k}(t,\cdot)-F_{J_k}(t,\cdot)}{\Delta}\right)dt\\
\geq&\int_0^T\liminf_{k\to\infty}\chi_{\{t\leq T-\Delta_k\}}L_s^{\delta_k}\left(\boldsymbol{r}_{J_k}(t),\frac{\boldsymbol{r}^{J_k}(t)-\boldsymbol{r}_{J_k}(t,\cdot)}{\Delta},\frac{F^{J_k}(t,\cdot)-F_{J_k}(t,\cdot)}{\Delta}\right)dt\\
\geq&\int_0^T L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))dt.
\end{aligned}$$

So we have established the lower semicontinuity of $I_s(\boldsymbol{r},\nu)$.

The lower semicontinuity of $I_f(\boldsymbol{r},\nu)$ can be done similarly. Recall the definition of $S^\delta(\boldsymbol{z},\boldsymbol{w})$ in Equation (3.18). We have that $S^\delta(\boldsymbol{z},\boldsymbol{w})$ is lower semicontinuous in δ, $\boldsymbol{z}$, and $\boldsymbol{w}$ and convex in $\boldsymbol{w}$ by lemmas A.1 and A.2. With exactly the procedure as with proving the lower semicontinuity of $I_s(\boldsymbol{r},\nu)$, we can establish

$$\liminf_{n\to\infty}\int_0^T S(\boldsymbol{r}_n(t),n_{\nu_n}(t,\cdot))dt \geq \int_0^{T-\Delta} S^\delta\left(\boldsymbol{r}_J(t),\frac{F^J(t,\cdot)-F_J(t,\cdot)}{\Delta}\right)dt$$

for a fine enough partition. Again, we consider the sequence of nested partition $J_k=2^k$ and $\Delta_k=T/2^k$. By Fatou's Lemma and the lower semicontinuity of S,

$$\begin{aligned}
&\liminf_{k\to\infty}\int_0^{T-\Delta_k} S^{\delta_k}\left(\boldsymbol{r}_{J_k}(t),\frac{F^{J_k}(t,\cdot)-F_{J_k}(t,\cdot)}{\Delta}\right)dt\\
&\geq \int_0^T \liminf_{k\to\infty}\chi_{\{t\leq T-\Delta_k\}}S^{\delta_k}\left(\boldsymbol{r}_{J_k}(t),\frac{F^{J_k}(t,\cdot)-F_{J_k}(t,\cdot)}{\Delta}\right)dt\\
&\geq \int_0^T S(\boldsymbol{r}(t),n_\nu(t,\cdot))dt.
\end{aligned}$$

Thus, we obtain the lower semicontinuity of $I_f(\boldsymbol{r},\nu)$. The proof of Equations (A.6) and (A.7) are similar. □

Lemma A.9. *Given $\boldsymbol{r}\in\mathbb{D}^d[0,T]$, $\nu\in\mathbb{M}_L[0,T]$, and $\epsilon>0$, there exist step functions $\boldsymbol{\theta}(t)\in R^d$ and $\boldsymbol{\alpha}(t)\in R^D$ such that*

$$I_s(\boldsymbol{r},\nu,\boldsymbol{\theta})\geq I_s(\boldsymbol{r},\nu)-\epsilon, \tag{A.9}$$

$$I_s^\delta(\boldsymbol{r},\nu,\boldsymbol{\theta})\geq I_s^\delta(\boldsymbol{r},\nu)-\epsilon, \tag{A.10}$$

and

$$I_f(\boldsymbol{r},\nu,\boldsymbol{\alpha})\geq I_f(\boldsymbol{r},\nu)-\epsilon, \tag{A.11}$$

$$I_f^\delta(\boldsymbol{r},\nu,\boldsymbol{\alpha})\geq I_f^\delta(\boldsymbol{r},\nu)-\epsilon. \tag{A.12}$$

The proof of Equations (A.9) and (A.10) can be referred to Lemma 5.43 in [26] and the proof of Equations (A.11) and (A.12) is similar. We will outline the main procedure here.

Proof. First, we consider Equation (A.9). If $\boldsymbol{r}$ is not absolutely continuous, $I_s(\boldsymbol{r},\nu,\boldsymbol{\alpha})=\infty$ by definition, so nothing needs to be proved. Now let us consider the case that $\boldsymbol{r}$ is absolutely continuous. For convenience, let $L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p}):=\langle\boldsymbol{p},\boldsymbol{\beta}\rangle-H_s(\boldsymbol{z},\boldsymbol{p},\boldsymbol{w})$. Since by definition $L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p})\leq L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$ for any $\boldsymbol{p}$, we have for B large enough

$$\begin{aligned}
&\int_0^T \chi_{\{\|\dot{\boldsymbol{r}}(t)\|\geq B\}}L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot),\boldsymbol{\theta}(t))dt\\
&\leq \int_0^T \chi_{\{\|\dot{\boldsymbol{r}}(t)\|\geq B\}}L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))dt\\
&\leq \epsilon/4.
\end{aligned}$$

by Lemma A.7. Choose $\boldsymbol{\theta}_1(t)=\mathbf{0}$ whenever $\|\dot{\boldsymbol{r}}(t)\|\geq B$ or $\dot{\boldsymbol{r}}(t)$ is not in $\mathcal{C}$ as defined in Equation (2.12). Let $R:=\sup_{0\leq t\leq T}\|\boldsymbol{r}(t)\|$. Since $\boldsymbol{r}$ is continuous, R is finite. Simply replacing λ_i by $\sum_{j=1}^{D}\lambda_i(\boldsymbol{z},j)w_j$ in Lemma 5.23 of [26], we have for B_1 large enough,

$$\sup_{|\boldsymbol{p}|\leq B_1} L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p})\geq L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})-\frac{\epsilon}{8T}$$

for all $\|\boldsymbol{z}\|\leq R$, $\|\boldsymbol{\beta}\|\leq B$ in $\mathcal{C}$ and $\boldsymbol{w}$ in Δ_D. So, for any $(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$ in bounded set

$$A:=\left\{\|\boldsymbol{z}\|\leq R,\boldsymbol{\beta}\in\mathcal{C},\|\boldsymbol{\beta}\|\leq B,\boldsymbol{w}\in\Delta_D\right\},$$

there exist a $\boldsymbol{p}_{\boldsymbol{z\beta w}}$ with $\|\boldsymbol{p}_{\boldsymbol{z\beta w}}\|\leq B_1$ such that

$$L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p}_{\boldsymbol{z\beta w}})\geq L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})-\frac{\epsilon}{4T}.$$

On the bounded set

$$\left\{\|\boldsymbol{z}\|\leq R,\boldsymbol{\beta}\in\mathcal{C},\|\boldsymbol{\beta}\|\leq B,\boldsymbol{w}\in\Delta_D,\|\boldsymbol{p}\|\leq B_1\right\},$$

the function $L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p})$ is uniformly continuous. What's more, by lemmas 5.22 and 5.33 in [26], $L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$ is continuous in A. So, given any $(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})\in A$, there exist a $\delta_{\boldsymbol{z\beta w}}>0$ such that

$$L_s(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{w}},\boldsymbol{p}_{\boldsymbol{z\beta w}})\geq L_s(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{w}})-\frac{\epsilon}{2T}$$

holds for any $(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{\beta}},\tilde{})\in O_{\boldsymbol{z\beta w}}\cap A$, where $O_{\boldsymbol{z\beta w}}$ is the $\delta_{\boldsymbol{z\beta w}}$-neighborhood of $(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$. By the Heine–Borel theorem, we can choose finite number of $O_{\boldsymbol{z}^i\boldsymbol{\beta}^j\boldsymbol{w}^k}$ to cover A. This means that

$$L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w},\boldsymbol{p}_{\boldsymbol{z}^i\boldsymbol{\beta}^j\boldsymbol{w}^k})\geq L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})-\frac{\epsilon}{2T}$$

whenever $\|\boldsymbol{z}-\boldsymbol{z}^i\|+\|\boldsymbol{\beta}-\boldsymbol{\beta}^j\|+\|\boldsymbol{w}-\boldsymbol{w}^k\|\leq\delta_{\boldsymbol{z}^i\boldsymbol{\beta}^j\boldsymbol{w}^k}$.

Define the function $\boldsymbol{\theta}_1(t)=\boldsymbol{p}_{\boldsymbol{z}^i\boldsymbol{\beta}^j\boldsymbol{w}^k}$ whenever $\|\boldsymbol{r}(t)-\boldsymbol{z}^i\|+\|\dot{\boldsymbol{r}}(t)-\boldsymbol{\beta}^j\|+\|n_\nu(t,\cdot)-\boldsymbol{w}^k\|\leq\delta_{\boldsymbol{z}^i\boldsymbol{\beta}^j\boldsymbol{w}^k}$ with some tie-breaking rule. The function $\boldsymbol{\theta}_1(t)$ takes a finite number of values. It may not be constant on intervals of time. So we approximate $\boldsymbol{\theta}_1(t)$ by a step function. Choose η small enough such that

$$\int_0^T \chi_{\{t\in B\}}L_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_\nu(t,\cdot))dt\leq\frac{\epsilon}{4}$$

whenever the set B has measure less than η. Since $\boldsymbol{\theta}_1(t)$ is a simple function, we can approximate it by a step function $\boldsymbol{\theta}$, and it agrees with $\boldsymbol{\theta}_1$ outside of a set of measure η (c.f. [25]).We finish the proof for Equation (A.9) by collecting all approximations above.

For the proof for Equation (A.11), we take advantage of Lemma 5.23 in [26] again by replacing λ_i with $\sum_{j=1}^{D}w_iq_{ij}$. We have for C large enough

$$\sup_{\|\boldsymbol{\sigma}\|\leq C} S(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma})\geq S(\boldsymbol{z},\boldsymbol{w})-\frac{\epsilon}{4T}.$$

On the bounded set

$$\left\{\|\boldsymbol{\sigma}\|\leq C,\|\boldsymbol{z}\|\leq R,\boldsymbol{w}\in\Delta_D\right\},$$

the function $S(\boldsymbol{z},\boldsymbol{w},\boldsymbol{\sigma})$ is uniformly continuous. With the similar strategy for L_s, we can find the desired step function $\boldsymbol{\alpha}$. So we finish the proof for Equation (A.11). The proof for I_s^δ and I_f^δ are similar. ∎

A.2. Part 2. Proof of lemmas related to the lower bound estimate.

Proof. **(Proof of Lemma 3.9.)** Since $S(\boldsymbol{z},\boldsymbol{w})$ is bounded by $Q:=\sum_{i,j=1}^{D}\sup_{\boldsymbol{z}} q_{ij}(\boldsymbol{z})$, there exists $(\eta(s),\psi(s))\in\mathcal{S}$ for any $s\in[t_m,t_{m+1}]$ such that

$$\begin{aligned}&\sum_{i=1}^{D}\psi_i(s)\sum_{j=1}^{D}\Big(\eta_{ij}(s)\log\frac{\eta_{ij}(s)}{q_{ij}(\boldsymbol{y}(s))}+q_{ij}(\boldsymbol{y}(s))-\eta_{ij}(s)\Big)\\&\leq S(\boldsymbol{y}(s),n_\nu(s,\cdot))+\epsilon\end{aligned}$$

and

$$|\psi_i(s)-n_\nu(s,i)|<\epsilon/(8DT)$$

by Lemma 8.61 in [26]. For each fixed $s\in[t_m,t_{m+1}]$, there exists $\delta_s>0$ such that

$$\begin{aligned}&\sum_{i=1}^{D}\psi_i(s)\sum_{j=1}^{D}\Big(\eta_{ij}(s)\log\frac{\eta_{ij}(s)}{q_{ij}(\boldsymbol{y}(s))}+q_{ij}(\boldsymbol{y}(s))-\eta_{ij}(s)\Big)\\&\leq S(\boldsymbol{y}(t),n_\nu(t,\cdot))+2\epsilon\end{aligned}$$

and

$$|\psi_i(s)-n_\nu(t,i)|<\epsilon/(4DT)$$

hold for any $t\in O_s=(s-\delta_s,s+\delta_s)\cap[t_m,t_{m+1}]$. By the Heine-Borel theorem, we can choose finite number of O_{s_k} in $\{O_s\}_{s\in[t_m,t_{m+1}]}$ to cover $[t_m,t_{m+1}]$. This means that there exists a further subdivision of interval $[t_m,t_{m+1}]$ (i.e. $t_m=t_{m0}<t_{m1}<\cdots<t_{mK_m}=t_{m+1}$) and related $(\psi^m(s_k),\eta^m(s_k))\in\mathcal{S}$ such that, for all $t\in[t_{mk},t_{m,k+1}]$,

$$\begin{aligned}&\sum_{i=1}^{D}\psi_i^m(s_k)\sum_{j=1}^{D}\Big(\eta_{ij}^m(s_k)\log\frac{\eta_{ij}^m(s_k)}{q_{ij}(\boldsymbol{y}(t))}+q_{ij}(\boldsymbol{y}(t))-\eta_{ij}^m(s_k)\Big)\\&\leq S(\boldsymbol{y}(t),n_\nu(t,\cdot))+2\epsilon.\end{aligned}$$

Since $\log q_{ij}(\boldsymbol{z})$ are bounded and Lipschitz continuous in $\boldsymbol{z}$, we can establish that $S(\boldsymbol{z},\boldsymbol{w})$ is absolutely continuous in $\boldsymbol{z}$, and this absolute continuity is uniform in $\boldsymbol{w}\in\Delta_D$. To see this, we only need to show that the function

$$f(\boldsymbol{x},\boldsymbol{w}):=\sup_{\boldsymbol{\sigma}\in\mathbb{R}^D}\left(-\sum_{i,j=1}^{D}w_ix_{ij}\left(e^{\langle\boldsymbol{\sigma},\boldsymbol{e}_{ij}\rangle}-1\right)\right)$$

is absolutely continuous in $\boldsymbol{x}=(x_{11},x_{12},\ldots,x_{DD})\in[1/\Lambda,\Lambda]^{D^2}$ (as defined in Equation (2.11)), uniformly in $\boldsymbol{w}\in\Delta_D$. For any $\boldsymbol{x},\boldsymbol{x}+\Delta\boldsymbol{x}\in[1/\Lambda,\Lambda]^{D^2}$ with $\|\Delta\boldsymbol{x}\|\leq1/4\Lambda$, let $h=1/4\Lambda$ and $r=\|\Delta\boldsymbol{x}\|/(h+\|\Delta\boldsymbol{x}\|)$, and define $\boldsymbol{q}=\boldsymbol{x}+\Delta\boldsymbol{x}/r$. With this construction, we have $\boldsymbol{q}\in[1/2\Lambda,M+1/2\Lambda]^{D^2}$, $f(\boldsymbol{x},\boldsymbol{w})$, $f(\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{w})$, $f(\boldsymbol{q},\boldsymbol{w})\in[0,(\Lambda+1/2\Lambda)D^2]$, and $\boldsymbol{x}+\Delta\boldsymbol{x}=(1-r)\boldsymbol{x}+r\boldsymbol{q}$. From the convexity of $f(\boldsymbol{x},\boldsymbol{w})$ in $\boldsymbol{x}$, we have

$$f(\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{w})\leq(1-r)f(\boldsymbol{x},\boldsymbol{w})+rf(\boldsymbol{q},\boldsymbol{w}),$$

and thus

$$f(\boldsymbol{x}+\Delta\boldsymbol{x},\boldsymbol{w})-f(\boldsymbol{x},\boldsymbol{w})\leq r(f(\boldsymbol{q},\boldsymbol{w})-f(\boldsymbol{x},\boldsymbol{w}))\leq4\Lambda\Big(\Lambda+\frac{1}{2\Lambda}\Big)D^2\|\Delta\boldsymbol{x}\|.$$

The absolute continuity in $\boldsymbol{z}$ and uniformity in $\boldsymbol{w}$ of $S(\boldsymbol{z},\boldsymbol{w})$ ensures that the estimate

$$S(\boldsymbol{y}(t),n_\nu(t,\cdot))\le S(\boldsymbol{r}(t),n_\nu(t,\cdot))+\epsilon$$

holds when J is large enough.

To simplify the notation, we will rewrite $\eta^m(s_k)$ as η^{mk} and $\psi^m(s_k)$ as ψ^{mk}. So, for each m, we have

$$\sum_{k=0}^{K_m-1}\int_{t_{mk}}^{t_{m,k+1}}\sum_{i=1}^{D}\psi_i^{mk}\sum_{j=1}^{D}\Big(\eta_{ij}^{mk}\log\frac{\eta_{ij}^{mk}}{q_{ij}(\boldsymbol{y}(t))}+q_{ij}(\boldsymbol{y}(t))-\eta_{ij}^{mk}\Big)dt$$
$$\le\int_{t_m}^{t_{m+1}}S(\boldsymbol{r}(t),n_\nu(t,\cdot))dt+3(t_{m+1}-t_m)\epsilon.$$

The proof is completed. ∎

Proof. **(Proof of Lemma 3.10.)** Define

$$\tilde f(\boldsymbol{\mu},\lambda(\boldsymbol{z},\cdot),\boldsymbol{w}):=\sum_{i=1}^{S}\Big(\sum_{j=1}^{D}\lambda_i(\boldsymbol{z},j)w_j-\mu_i+\mu_i\log\frac{\mu_i}{\sum_{j=1}^D\lambda_i(\boldsymbol{z},j)w_j}\Big)$$

and

$$\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})=\inf_{\boldsymbol{\mu}\in\mathcal{K}_{\boldsymbol{\beta}}}\tilde f(\boldsymbol{\mu},\lambda(\boldsymbol{z},\cdot),\boldsymbol{w}).$$

Taking advantage of Theorem 5.26 of [26], we have

$$\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})=L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w}).\tag{A.13}$$

We will show that, for any B_1, $\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$ is continuous in $\boldsymbol{z}$ and $\boldsymbol{w}$, uniformly in $\boldsymbol{\beta}$ in

$$\mathcal{V}:=\{\boldsymbol{\beta}\in\mathcal{C},\|\boldsymbol{\beta}\|\le B_1\},$$

where $\mathcal{C}$ is the cone defined in Equation (2.12).

By Lemma 5.20 of [26], we can find a constant B_2 such that, for any $\boldsymbol{\beta}\in\mathcal{V}$ there exists a $\boldsymbol{\mu}\in\mathcal{K}_{\boldsymbol{\beta}}$ with $\|\boldsymbol{\mu}\|\le B_2$. Therefore, for all $\boldsymbol{\beta}\in\mathcal{V}$ and any $\boldsymbol{\mu}\in\mathcal{K}_{\boldsymbol{\beta}}$ with $\|\boldsymbol{\mu}\|\le B_2$,

$$\begin{aligned}&\tilde L_s(\boldsymbol{z}',\boldsymbol{\beta},\boldsymbol{w}')-\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})\\ \le&\tilde f(\boldsymbol{\mu},\lambda(\boldsymbol{z}',\cdot),\boldsymbol{w}')-\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})\\ \le&\tilde f(\boldsymbol{\mu},\lambda(\boldsymbol{z},\cdot),\boldsymbol{w}')-\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})+C_1\|\boldsymbol{z}'-\boldsymbol{z}\|\\ \le&\tilde f(\boldsymbol{\mu},\lambda(\boldsymbol{z},\cdot),\boldsymbol{w})-\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})+C_1\|\boldsymbol{z}'-\boldsymbol{z}\|+C_2\|\boldsymbol{w}'-\boldsymbol{w}\|\end{aligned}$$

for some positive constants C_1 and C_2. Now choose $\boldsymbol{\mu}$ to minimize $\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})$ to establish that

$$\tilde L_s(\boldsymbol{z}',\boldsymbol{\beta},\boldsymbol{w}')-\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})\le C_1\|\boldsymbol{z}'-\boldsymbol{z}\|+C_2\|\boldsymbol{w}'-\boldsymbol{w}\|.\tag{A.14}$$

By Lemma 5.17 and Lemma 5.32 of [26] (replacing $\lambda_i(\boldsymbol{x})$ with $\sum_{j=1}^D\lambda_i(\boldsymbol{x},j)w_j$), we know that there exist positive constants M_1, M_2, and B so that, for all $\boldsymbol{\beta}\in\mathcal{C}$ with $\|\boldsymbol{\beta}\|\ge B$, for all $\boldsymbol{z}\in\mathbb{R}^d$, and all $\boldsymbol{w}\in\Delta_D$,

$$M_1\|\boldsymbol{\beta}\|\log\|\boldsymbol{\beta}\|\le\tilde L_s(\boldsymbol{z},\boldsymbol{\beta},\boldsymbol{w})\le M_2\|\boldsymbol{\beta}\|\log\|\boldsymbol{\beta}\|.$$

So, for any $\boldsymbol{q}\in\mathbb{D}^d[0,T]$ and any $\tilde{\nu}\in\mathbb{M}[0,T]$,

$$\begin{aligned}&\int_0^T \chi_{\{\|\dot{\boldsymbol{r}}(t)\|\geq B\}}\tilde{L}_s(\boldsymbol{q}(t),\dot{\boldsymbol{r}}(t),n_{\tilde{\nu}}(t,\cdot))dt\\ \leq&\int_0^T \chi_{\{\|\dot{\boldsymbol{r}}(t)\|\geq B\}}M_2\|\dot{\boldsymbol{r}}(t)\|\log\|\dot{\boldsymbol{r}}(t)\|dt\\ \leq&\int_0^T \chi_{\{\|\dot{\boldsymbol{r}}(t)\|\geq B\}}\frac{M_2}{M_1}\tilde{L}_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_{\nu}(t,\cdot))dt\\ :=&\epsilon(B).\end{aligned}\tag{A.15}$$

By Lemma A.7, we have that, $\epsilon(B)\to 0$ as $B\to\infty$. Combining Equations (A.14) and (A.15), we have for any $\epsilon>0$, there exists a $\delta>0$ such that

$$\sup_{0\leq t\leq T}\|\boldsymbol{q}(t)-\boldsymbol{r}(t)\|<\delta \quad\text{and}\quad \sup_{0\leq t\leq T}\|n_{\tilde{\nu}}(t,\cdot)-n_{\nu}(t,\cdot)\|<\delta$$

implies

$$\left|\int_0^T \tilde{L}_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_{\nu}(t,\cdot))dt-\int_0^T \tilde{L}_s(\boldsymbol{q}(t),\dot{\boldsymbol{r}}(t),n_{\tilde{\nu}}(t,\cdot))dt\right|\leq\epsilon.\tag{A.16}$$

With this continuity property, we have

$$\begin{aligned}&\int_0^T \tilde{L}_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_{\nu}(t,\cdot))dt\\ =&\sum_{m=0}^{J-1}\int_{t_m}^{t_{m+1}}\tilde{L}_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_{\nu}(t,\cdot))dt\\ \geq&\sum_{m=0}^{J-1}\int_{t_m}^{t_{m+1}}\tilde{L}_s(\boldsymbol{r}(t_m),\dot{\boldsymbol{r}}(t),n_{\pi}(t_m,\cdot))dt-\epsilon\\ \geq&\sum_{m=0}^{J-1}\Delta\cdot\tilde{L}_s\left(\boldsymbol{r}(t_m),\frac{\boldsymbol{r}(t_{m+1})-\boldsymbol{r}(t_m)}{\Delta},n_{\pi}(t_m,\cdot)\right)-\epsilon.\end{aligned}$$

By the definition of $\tilde{L}_s$, for each m, we have $\boldsymbol{\mu}^m\in K_{\boldsymbol{\beta}_m}$ such that

$$\begin{aligned}&\tilde{L}_s\left(\boldsymbol{r}(t_m),\frac{\boldsymbol{r}(t_{m+1})-\boldsymbol{r}(t_m)}{\Delta},n_{\pi}(t_m,\cdot)\right)\\ \geq&\sum_{i=1}^{S}\left(\lambda_i^{\pi}(\boldsymbol{y}(t_m))-\mu_i^m+\mu_i^m\log\frac{\mu_i^m}{\lambda_i^{\pi}(\boldsymbol{y}(t_m))}\right)-\epsilon/T,\end{aligned}$$

and finally we have

$$\begin{aligned}&\int_0^T \tilde{L}_s(\boldsymbol{r}(t),\dot{\boldsymbol{r}}(t),n_{\nu}(t,\cdot))dt\\ \geq&\sum_{m=0}^{J-1}\Delta\cdot\sum_{i=1}^{S}\left(\lambda_i^{\pi}(\boldsymbol{y}(t_m))-\mu_i^m+\mu_i^m\log\frac{\mu_i^m}{\lambda_i^{\pi}(\boldsymbol{y}(t_m))}\right)-2\epsilon\end{aligned}$$

$$\geq \int_0^T \sum_{i=1}^S \left(\lambda_i^\pi(\boldsymbol{y}(t)) - \mu_i^m + \mu_i^m \log \frac{\mu_i^m}{\lambda_i^\pi(\boldsymbol{y}(t))}\right) dt - 3\epsilon. \tag{A.17}$$

Lemma 3.10 is proved by combing (A.13) and (A.17). ∎

Proof. **(Proof of Lemma 3.11.)** We need to prove that, for any bounded continuous function $h(t,z)$,

$$\lim_{n\to\infty} \int_0^T h(t,\bar{\xi}_n(t))dt = \int_0^T \sum_{i=1}^D h(t,i) n_\pi(t,i) dt$$

in probability. It suffices to prove that for each time interval $[t_{mk}, t_{m,k+1}]$,

$$\lim_{n\to\infty} \int_{t_{mk}}^{t_{m,k+1}} h(t,\bar{\xi}_n(t))dt = \int_{t_{mk}}^{t_{m,k+1}} \sum_{i=1}^D h(t,i) n_\pi(t,i) dt.$$

Since $\bar{\xi}_n$ lives on only finite states, for any $\epsilon > 0$, there exists $\delta > 0$ such that for $|t_k - t| < \delta$

$$|h(t,\bar{\xi}_n(t)) - h(t_k,\bar{\xi}_n(t))| < \epsilon$$

for all $t \in [t_{mk}, t_{m,k+1}]$.

Take an integer L large enough and define $\tilde{\delta} = (t_{m,k+1} - t_{mk})/L < \delta$. Let $\tau_l = t_{mk} + l\tilde{\delta}$ for $l = 0,1,\ldots,L$. We then have

$$\begin{aligned}
\limsup_{n\to\infty} \int_{t_{mk}}^{t_{m,k+1}} h(t,\bar{\xi}_n(t))dt &= \limsup_{n\to\infty} \sum_{l=0}^{N-1} \int_{\tau_l}^{\tau_{l+1}} h(t,\bar{\xi}_n(t))dt \\
&\leq \limsup_{n\to\infty} \sum_{l=0}^{N-1} \int_{\tau_l}^{\tau_{l+1}} h(\tau_l,\bar{\xi}_n(t))dt + T\epsilon \\
&= \sum_{k=0}^{N-1} \int_{\tau_l}^{\tau_{l+1}} \sum_{i=1}^D h(\tau_l,i) n_\pi(t,i) dt + T\epsilon \\
&\leq \sum_{k=0}^{N-1} \int_{\tau_l}^{\tau_{l+1}} \sum_{i=1}^D h(t,i) n_\pi(t,i) dt + 2T\epsilon \\
&= \int_{t_{mk}}^{t_{m,k+1}} \sum_{i=1}^D h(t,i) n_\pi(t,i) dt + 2T\epsilon.
\end{aligned} \tag{A.18}$$

In Equation (A.18), we utilized the ergodicity of the process $\bar{\xi}_n$ on each interval $[\tau_l, \tau_{l+1})$. The convergence can be obtained in the almost sure and $L^1_{\mathbb{P}}$-sense rather than in probability [6]. Similarly, we can prove

$$\liminf_{n\to\infty} \int_0^T h(t,\bar{\xi}_n(t))dt \geq \int_{t_{mk}}^{t_{m,k+1}} \sum_{i=1}^D h(t,i) n_\pi(t,i) dt - 2T\epsilon.$$

The proof is completed. ∎

Proof. **(Proof of Lemma 3.12.)** The goal is to prove that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} \|\bar{\boldsymbol{z}}_n(t) - \boldsymbol{y}(t)\| \geq \epsilon\right) = 0.$$

For any $\boldsymbol{p}\in\mathbb{R}^d$ and $\rho>0$, we have the martingale

$$\begin{aligned}M_t=&\exp\Bigg\{\langle\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t),\rho\boldsymbol{p}\rangle\\&-\int_0^t\sum_{i=1}^S\Big(n\mu_i(s)\frac{\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))}{\lambda_i^\pi(\boldsymbol{y}(s))}(e^{\langle\rho\boldsymbol{p},\boldsymbol{u}_i/n\rangle}-1)-\mu_i(s)\langle\rho\boldsymbol{p},\boldsymbol{u}_i\rangle\Big)ds\Bigg\}\\=&\exp\Bigg\{\langle\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t),\rho\boldsymbol{p}\rangle\\&-\int_0^t\sum_{i=1}^S\Big(\mu_i(s)\frac{\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))-\lambda_i^\pi(\boldsymbol{y}(s))}{\lambda_i^\pi(\boldsymbol{y}(s))}\langle\rho\boldsymbol{p},\boldsymbol{u}_i\rangle\\&+\mu_i(s)\frac{\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))}{\lambda_i^\pi(\boldsymbol{y}(s))}\big(n(e^{\langle\rho\boldsymbol{p},\boldsymbol{u}_i/n\rangle}-1)-\langle\rho\boldsymbol{p},\boldsymbol{u}_i\rangle\big)\Big)ds\Bigg\}.\end{aligned}$$

Recall that Assumption 2.2 and $\mu_i(t)$ is piecewise constant and bounded. We can perform similar estimate as in Lemma A.5 to obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T}\left\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)-\int_0^t\sum_{i=1}^S\mu_i(s)\frac{\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))-\lambda_i^\pi(\boldsymbol{y}(s))}{\lambda_i^\pi(\boldsymbol{y}(s))}ds\boldsymbol{u}_i\right\|\ge\epsilon\right)\\&\le\exp\Big(-n\epsilon c_1\log(\epsilon c_2)\Big),\end{aligned}\tag{A.19}$$

where c_1 and c_2 are positive constants. By Lemma 3.11, we have

$$\begin{aligned}&\int_0^t\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))ds-\int_0^t\lambda_i^\pi(\boldsymbol{y}(s))ds\\&=\left(\int_0^t\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))ds-\int_0^t\lambda_i(\boldsymbol{y}(s),\bar{\xi}_n(s))ds\right)\\&\quad+\left(\int_0^t\sum_{j=1}^D\lambda_i(\boldsymbol{y}(s),j)n_{\bar{\nu}_n}(s,j)ds-\int_0^t\lambda_i^\pi(\boldsymbol{y}(s))ds\right)\\&\leqslant K\int_0^t\|\bar{\boldsymbol{z}}_n(s)-\boldsymbol{y}(s)\|ds+B_n,\end{aligned}\tag{A.20}$$

where

$$B_n=\sup_{t\in[0,T]}\left|\sum_{j=1}^D\int_0^t\lambda_i(\boldsymbol{y}(s),j)\Big(n_{\bar{\nu}_n}(s,j)-n_\pi(s,j)\Big)ds\right|\to0\tag{A.21}$$

as n goes to infinity for $t\le T$.

Define $C=dAUT\Lambda$, where

$$A=\max_{t\in[0,T]}\max_{i=1,\dots,S}\mu_i(t)\quad\text{and}\quad U=\max_{i=1,\dots,S}\|\boldsymbol{u}_i\|.$$

Combining Equations (A.19), (A.20), and (2.11), we have

$$\mathbb{P}\left(\sup_{0\le t\le T}\Big(\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)\|-CK\int_0^t\|\bar{\boldsymbol{z}}_n(s)-\boldsymbol{y}(s)\|ds-CB_n\Big)\ge\epsilon\right)$$

$$\leq \mathbb{P}\left(\sup_{0\leq t\leq T}\left\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)-\int_0^t\sum_{i=1}^S \mu_i(t)\frac{\lambda_i(\bar{\boldsymbol{z}}_n(s),\bar{\xi}_n(s))-\lambda_i^\pi(\boldsymbol{y}(s))}{\lambda_i^\pi(\boldsymbol{y}(s))}ds\boldsymbol{u}_i\right\|\geq\epsilon\right)$$
$$\leq \exp\Big(-n\epsilon c_1\log(\epsilon c_2)\Big). \tag{A.22}$$

From Equation (A.22) and Gronwall's inequality, we obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)\|\geqslant(\epsilon+CB_n)e^{CKT}\right)\\ \leqslant&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\left(\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)\|-CK\int_0^t\|\bar{\boldsymbol{z}}_n(s)-\boldsymbol{y}(s)\|ds\right)\geqslant\epsilon\right)\\ \leqslant&\exp\Big(-n\epsilon c_1\log(\epsilon c_2)\Big).\end{aligned}$$

Combing the condition (A.21) and the inequality

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)\|\geqslant 2\epsilon e^{CKT}\right)\\ \leq&\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}\|\bar{\boldsymbol{z}}_n(t)-\boldsymbol{y}(t)\|\geqslant(\epsilon+CB_n)e^{CKT}\right)+\mathbb{P}\left(B_n\geqslant\frac{\epsilon}{C}\right),\end{aligned}$$

we finish the proof. ∎

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REFERENCES

[1] M. Assaf, E. Roberts, and Z. Luthey-Schulten, *Determining the Stability of Genetic Switches: Explicitly Accounting for mRNA Noise*, Phys. Rev. Lett., 106, 2048102, 2011.
[2] P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1999.
[3] M. Chen and Y. Lu, *On evaluating the rate functions of large deviations for jump processes*, Acta. Math. Sin.. 6, 206–219, 2015.
[4] Y. Chen, C. Lv, F. Li, and T. Li, *Distinguishing the rates of gene activation from phenotypic variations*, BMC Syst. Biol., 9, 29, 2015.
[5] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer-Verlag, New York, Second Edition, 1998.
[6] R. Durrett, *Probability: Theory and Examples*, Forth Edition, Cambridge University Press, Cambridge, 2010.
[7] W. E, D. Liu, and E. Vanden-Eijnden, *Analysis of multiscale methods for stochastic differential equations*, Comm. Pure App. Math., 58, 1544–1585, 2005.
[8] W. E, D. Liu, and E. Vanden-Eijnden, *Nested stochastic simulation algorithms for chemical kinetic systems with multiple time scales*, J. Comp. Phys., 221, 158–180, 2007.
[9] R.S. Ellis, *Entropy, Large Deviations and Statistical Mechanics*, Springer-Verlag, New York, 1985.
[10] M. Elowitz, A. Levine, E. Siggia, and P. Swain, *Stochastic gene expression in a single cell*, Science, 297, 1183–1186, 2002.
[11] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence*, John Wiley and Sons, New York, 1986.
[12] M.I. Freidlin and A.D. Wentzel, *Random Perturbations of Dynamical Systems*, Springer-Verlag, New York, Second Edition, 1998.
[13] D.T. Gillespie, *Exact stochastic simulation of coupled chemical reactions*, J. Phys. Chem., 81, 2340–2361, 1977.
[14] D.T. Gillespie, *The chemical Langevin equation*, J. Chem. Phys., 113, 297–306, 2000.
[15] M. Heymann and E. Vanden-Eijnden, *The geometric minimum action method: A least action principle on the space of curves*, Comm. Pure Appl. Math., 61, 1052–1117, 2008.

[16] H. Kang and T.G. Kurtz, *Separation of time-scales and model reduction for stochastic reaction networks*, Ann. Appl. Probab., 23, 529–583, 2013.
[17] T.G. Kurtz, *The relationship between stochastic and deterministic models for chemical reactions*, J. Chem. Phys., 57, 2976–2978, 1972.
[18] T.G. Kurtz, *A limit theorem for perturbed operator semigroups with applications to random evolutions*, J. Funct. Anal., 12, 55–67, 1973.
[19] T. Li and F. Lin, *Two-scale large deviations for chemical reaction kinetics through second quantization path integral*, J. Phys. A: Math. Theor., 49, 135204, 2016.
[20] R. Liptser, *Large deviations for two scaled diffusions* , Prob. Theory Relat. Fields, 106, 71–104, 1996.
[21] C. Lv, X. Li, F. Li, and T. Li, *Constructing the energy landscape for genetic switching system driven by intrinsic noise*, PLoS ONE, 9, e88167, 2014.
[22] C. Lv, X. Li, F. Li, and T. Li, *Energy landscape reveals that the budding yeast cell cycle is a robust and adaptive multi-stage process*, PLoS Comput. Biol., 11 , e1004156, 2015.
[23] G. Papanicolaou, *Introduction to the Asymptotic Analysis of Stochastic Equations*, Lectures in Mathematics, American Mathematical Society, Rode Island, 16, 1977.
[24] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
[25] H.L. Royden, *Real Analysis*, Macmillan, New York, Second Edition, 1968.
[26] A. Shwartz and A. Weiss, *Large Deviations for Performance Analysis: Queues, Communications and Computing*, Chapman and Hall, London, 1995.
[27] A. Shwartz and A. Weiss, *Large deviations with diminishing rates*, Math. Oper. Res., 30, 281–310, 2005.
[28] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, Elsevier, Amsterdam, Third Edition, 2007.
[29] A.Yu. Veretennikov, *On large deviations for SDEs with small diffusion and averaging*, Stoch. Process. Appl., 89, 69–79, 2000.
[30] A.Yu. Veretennikov, *On large deviations in the averaging principle for SDE's with a "full dependence"*, Ann. Prob., 27, 284–296, 1999.
[31] K. Zhang, M. Sasai, and J. Wang, *Eddy current and coupled landscapes for nonadiabatic and nonequilibrium complex system dynamics*, Proc. Nat. Acad Sci. USA, 110, 14930–14935, 2013.
[32] P. Zhou and T. Li, *Construction of the landscape for multi-stable systems: potential landscape, quasi-potenital, A-type integral and beyond*, J. Chem. Phys., 144, 94109, 2016.