COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
Vol. 29, Nos. 5 & 6, pp. 903–923, 2004

Local Existence for the Dumbbell Model of Polymeric Fluids

Tiejun Li, Hui Zhang,* and Pingwen Zhang

LMAM and School of Mathematical Sciences,
Peking University, Beijing, P.R. China

ABSTRACT

A local existence and uniqueness theorem is proved for a micro-macro model for polymeric fluid, as well as the property of the solution. The polymer stress tensor is given by an integral which involves the solution of a diffusion equation, the coefficient of this diffusion equation depends on the gradient of the solution of the Navier–Stokes equation.

Key Words: Dumbbell model; Navier–Stokes equation; Fokker–Plank equation.

AMS Subject Classification: 76A05; 76D03; 35Q35.

1. INTRODUCTION

In this paper we prove the well-posedness of coupled kinetic–hydrodynamic models for polymeric fluids. These models differ from traditional hydrodynamic models by taking explicitly into account the micromechanical structure of the polymers. The simplest class of micromechanical models which is capable of reproducing some aspects of polymeric flow behavior is the class of dumbbell

*Correspondence: Hui Zhang, LMAM and School of Mathematical Sciences, Peking University, Yiheyun Road 5, Haidian District, Beijing 100871, P.R. China; Fax: 86-10-62767146; E-mail: huizhang@math.pku.edu.cn.

DOI: 10.1081/PDE-120037336
0360-5302 (Print); 1532-4133 (Online)

models. In these models, each polymer molecule is represented as two beads joined together by elastic spring or rigid rod. For dilute polymer solutions, elastic dumbbell models have been exclusively used for complex flow simulations. The configuration of the spring then specifies the conformation of the polymer.

An incompressible fluid is subject to the following system of equations:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \triangle \mathbf{u} + \nabla \cdot \tau, \quad \text{for } \mathbf{x} \in \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{for } \mathbf{x} \in \Omega, \tag{2}$$

$$\mathbf{u} = 0, \quad \text{for } \mathbf{x} \in \Gamma = \partial\Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary. Here p and $\mathbf{u}$ denote the pressure and the velocity field, respectively. All of the coefficients are normalized to one in the equations. In contrast to traditional models of complex fluids which express polymer stress τ using empirical constitutive relations, τ expresses the polymer stress in terms of the microscopic conformations of the polymers

$$\tau = \langle \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q} \rangle = \int_{\mathbb{R}^3} \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q}\psi \, d\mathbf{Q}, \tag{4}$$

where ψ obeys the Fokker–Planck equation

$$\psi_t + (\mathbf{u} \cdot \nabla)\psi = -\nabla_{\mathbf{Q}} \cdot [(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\psi] + \triangle_{\mathbf{Q}}\psi. \tag{5}$$

Here we use subscript $\mathbf{Q}$ to denote the derivatives with respect to $\mathbf{Q}$, without the subscript, the differentiation is understood to be in $\mathbf{x}$. $\mathbf{Q}$ is the independent variable in the configuration space of the dumbbell and ψ (depending on $\mathbf{x}, \mathbf{Q}$, and t) is configuration distribution function. $\kappa = (\nabla \mathbf{u})^T$, $\mathbf{g}(\mathbf{Q})$ is a vector function, which denotes the spring force between two beads. We refer the reader to Bird et al. (1987), Doi and Edwards (1986), and Risken (1984) for the details.

An elastic dumbbell model has six configuration degrees of freedom. Two types of elastic dumbbells are important in flow modeling. The first is the Hookean dumbbell. In this model, the connector is a Hookean spring, and the connector force $\mathbf{g}(\mathbf{Q})$ is given by $\mathbf{g}(\mathbf{Q}) = \mathbf{Q}$. In this special case, we get from (1)–(5) a reduced system of equations for $\mathbf{u}$ and τ,

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0, \tag{6}$$

$$\tau_t + (\mathbf{u} \cdot \nabla)\tau - (\nabla \mathbf{u})^T \tau - \tau \nabla \mathbf{u} + \tau - I = 0. \tag{7}$$

In this way, one eliminates $\mathbf{Q}$ as an independent variable. This is the well-known Oldroyd-B model. Its well-posedness has been studied by Guillopé and Saut (1990), Lions and Masmoudi (2000) and its numerical simulation has been completed by Feigl et al. (1995) and Laso and Öttinger (1993). However, their methods do not seem to extend to more general cases when closed systems of equations such as (6)–(7) are not available. The second type of model is done by using the force law

$$\mathbf{g}(\mathbf{Q}) = \frac{\mathbf{Q}}{1 - (Q/Q_0)^2}$$

where Q, the length of the connector, is not allowed to exceed some fixed Q_0. This is known as the FENE model, for Finitely Extensible Non-linear Elastic dumbbell.

In general, the spring force law should be nonlinear vector function. Assume the nonlinear force

$$\mathbf{g}(\mathbf{Q}) = f(|\mathbf{Q}|^2)\mathbf{Q} \tag{8}$$

where f satisfies the following condition:

(**G**) The function f is C^∞-smooth from $[0, \infty)$ to $(0, \infty)$, and there exist number $\sigma \geq 0$ and $k > 0$ such that $\lim_{|\mathbf{Q}|\to\infty} f(|\mathbf{Q}|^2)/|\mathbf{Q}|^\sigma = k$. Moreover,

$$\lim \sup_{|\mathbf{Q}|\to\infty} |f'(|\mathbf{Q}|^2)|/|\mathbf{Q}|^{\sigma-2} \leq C$$

and higher derivatives of f have at most polynomial growth as $|\mathbf{Q}| \to \infty$.

The recent resurgence of interest on (1)–(5) comes from chemical engineers who are interested in designing stochastic modeling techniques for polymeric fluids. One of the most popular approach uses the so-called Brownian configurational fields (BCF) which is a stochastic field variable $\mathbf{Q}(\mathbf{x}, t)$ that describes the local conformation of the polymers Hulsen et al. (1997). For the simplest dumbbell model, $\mathbf{Q}$ satisfies:

$$\mathbf{Q}_t + (\mathbf{u} \cdot \nabla)\mathbf{Q} = (\nabla\mathbf{u})^T\mathbf{Q} - \mathbf{g}(\mathbf{Q}) + \dot{\mathbf{v}}(t), \tag{9}$$

where $\dot{\mathbf{v}}(t)$ is the temporal Gaussian white noise. Equation (6) is the Fokker–Planck equation for (9). For smooth solutions the systems (1)–(5) and (1), (4), (9), are equivalent. For one-dimensional shear flows, the convergence numerical analysis was carried out in Li and Zhang (2002) and Jourdain et al. (2002) for linear spring force and well-posedness in Jourdain et al. (2004) for nonlinear spring force using the specific structure of the shear flow system. To extend such analysis to high dimension, a crucial step (Li and Zhang, 2003) has been done to analyze the local well-posedness of the coupled system (1), (4), (9) with periodic boundary condition.

A local existence and uniqueness theorem for solutions of Euler equation coupled with kinetic theory of polymeric fluid was proved by Renardy (1990, 1991). Energy methods were used to show the well-posedness of the Dirichlet initial boundary value problem for incompressible hypoelastic materials. Since ψ is a probability density, L^1 spaces are natural for the $\mathbf{Q}$ dependence. The framework is nice, but many important details for the estimates are missed out. These details are important for further analysis, in particular numerical analysis of the recently proposed multiscale numerical methods. In addition, the commonly used formulation of the dumbbell model (1)–(5) (Bird et al., 1987; Doi and Edwards, 1986; Guillopé and Saut, 1990; Jourdain et al., 2002; Lions and Masmoudi, 2000) is different from the one analyzed in Renardy (1991), where the model does not include a solvent viscosity. Therefore it is of interest to supplement the work of Renardy (1991) with a detailed well-posedness analysis for the (1)–(5). This is the main objective of the present paper.

Suppose the system is supplied with the initial value

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \psi(\mathbf{x}, \mathbf{Q}, 0) = \psi_0(\mathbf{x}, \mathbf{Q}). \tag{10}$$

and define the space for the distribution function ψ

$$\mathscr{X}_{n,0} = \left\{\psi : \mathbb{R}^3 \to \mathbb{R} \mid \int (1 + |\mathbf{Q}|^n)|\psi(\mathbf{Q})| d\mathbf{Q} < \infty\right\}, \tag{11}$$

moreover we let $\mathscr{X}_{n,k}$ be the space of all ψ whose derivatives with respect to $\mathbf{Q}$ up to order k lie in $\mathscr{X}_{n,0}$. Finally, set

$$\mathscr{X}_k = \bigcap_{n=0}^{\infty} \mathscr{X}_{n,k}, \tag{12}$$

with the natural topology of a Fréchet space. The space $\mathscr{X}_k$ consists of functions which, together with their derivatives, vanish at infinite order as $|\mathbf{Q}| \to \infty$. Next we state the main results.

The main assumptions of the theorem are:

(A1) The domain $\Omega \in \mathbb{R}^3$ is bounded and $\partial\Omega$ is of class C^4.
(A2) $\mathbf{u}_0 \in H^4(\Omega)$.
(A3) $\psi_0 \in H^4(\Omega, \mathscr{X}_2)$, where $H^k(\Omega, \mathscr{X}_l)$ stands for $\bigcap_{n=0}^{\infty} H^k(\Omega, \mathscr{X}_{n,l})$. Moreover, $\psi_0 \geq 0$ and $\int \psi_0(x, \mathbf{Q}) d\mathbf{Q} = 1$ for every $x \in \Omega$.

In addition, we need compatibility conditions between the initial data and the incompressibility and boundary conditions. So we assume the following compatibility conditions:

(C1) div $\mathbf{u}_0 = 0$ and $\mathbf{u}_0 = 0$ on $\partial\Omega$.
(C2) $\mathbf{u}_1$ vanishes on $\partial\Omega$.

where $\mathbf{u}_1 = \mathbf{u}_t|_{t=0}$.

Theorem 1. *Assume that* **(A1)–(A3)**, **(C1)–(C2)** *and* **(G)** *hold, then there exists* $T' > 0$ *such that the problem* (1)–(5) *and* (10) *has a unique solution with the regularity*

$$\mathbf{u} \in \bigcap_{k=0}^{2} H^k([0, T']; H^{4-2k}(\Omega)); \tag{13}$$

$$\tau \in \bigcap_{k=0}^{1} H^k([0, T']; H^{3-2k}(\Omega)); \tag{14}$$

$$\psi \in \bigcap_{k=0}^{1} H^k([0, T']; H^{3-2k}(\Omega, \mathscr{X}_0)). \tag{15}$$

The paper is organized as follows. In the Sec. 2, we define iterative scheme of the system to obtain the existence of the solution. The scheme alternates between solving an equation of the same type as encountered in incompressible elasticity and solving a linear diffusion equation. The Sec. 3 is devoted to giving the detailed proof the main lemmas, where the estimates of ψ with respect to Lagrangian variable are completed first, then the corresponding estimates with respect to Eulerian variable are obtained. We draw the conclusion in Sec. 4.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

2. ITERATIVE CONSTRUCTION OF SOLUTION

In this section we construct an iterative scheme of the system (1)–(3) and (5) to obtain the existence of the solution. Given an iterate $\mathbf{u}^m$ we determine $\mathbf{u}^{m+1}$ by solving the equations

$$\mathbf{u}_t^{m+1} + (\mathbf{u}^m \cdot \nabla)\mathbf{u}^{m+1} + \nabla p^{m+1} = \Delta \mathbf{u}^{m+1} + \nabla \cdot \tau^m, \tag{16}$$

$$\nabla \cdot \mathbf{u}^{m+1} = 0 \tag{17}$$

with the initial condition $\mathbf{u}^{m+1}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ and boundary condition $\mathbf{u}^{m+1}|_\Gamma = 0$, where

$$\tau^m = \int_{\mathbb{R}^3} \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q}\psi^m \, d\mathbf{Q}. \tag{18}$$

Meanwhile, for given $\mathbf{u}^m$, we determine ψ^{m+1} from the initial value problem

$$\psi_t^{m+1} + (\mathbf{u}^m \cdot \nabla)\psi^{m+1} = -\nabla_{\mathbf{Q}} \cdot [(\kappa^m \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\psi^{m+1}] + \triangle_{\mathbf{Q}} \psi^{m+1}, \tag{19}$$

$$\psi^{m+1}(\mathbf{x}, \mathbf{Q}, 0) = \psi_0(\mathbf{x}, \mathbf{Q}), \tag{20}$$

where $\kappa^m = (\nabla \mathbf{u}^m)^T$. Our eventual task is to show that the mapping $\mathcal{M} : \mathbf{u}^m \mapsto \mathbf{u}^{m+1}$ is a contraction in an appropriate complete space of functions. The fixed point of the mapping is the solution we seek.

We will consider the mapping $\mathcal{M}$ in the function space $S(M, T)$ with metric $d(\cdot, \cdot)$ showed next. $S(M, T)$ is the set of all functions $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^3$ with the following properties:

$$\mathbf{u} \in \bigcap_{k=0}^{2} H^k([0, T]; H^{4-2k}(\Omega)), \tag{21}$$

$$\|\mathbf{u}\|_{0,4} + \|\mathbf{u}\|_{1,2} + \|\mathbf{u}\|_{2,0} \le M, \tag{22}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{23}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma, \tag{24}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \mathbf{u}_t(\mathbf{x}, 0) = \mathbf{u}_1(\mathbf{x}). \tag{25}$$

Here $\| \cdot \|_{k,l}$ denotes the norm in $H^k([0, T]; H^l(\Omega))$. The function $\mathbf{u}_0, \mathbf{u}_1$ lie in $H^4(\Omega)$ and $H^2(\Omega)$, respectively. On $S(M, T)$, we define the metric

$$d(\mathbf{u}_1, \mathbf{u}_2) = \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,4} + \|\mathbf{u}_1 - \mathbf{u}_2\|_{1,2} + \|\mathbf{u}_1 - \mathbf{u}_2\|_{2,0}. \tag{26}$$

Lemma 1. *If M is chosen large enough, $S(M, T)$ is not empty. Moreover, the metric d is complete on $S(M, T)$.*

The proof will be given along the line of the argument of Lemma 1 by Renardy (1990) with slight modifications.

The contraction of the mapping $\mathscr{M}$ is established by proving the next five lemmas.

Lemma 2. *Assume that the bounds of the following kind hold*:

$$\|\mathbf{u}^m\|_{0,4} + \|\mathbf{u}^m\|_{1,2} + \|\mathbf{u}^m\|_{2,0} \leq M, \tag{27}$$

$$\|\tau^m\|_{0,3} + \|\tau^m\|_{1,1} \leq K \tag{28}$$

Then (16)–(18) *has a solution*

$$\mathbf{u}^{m+1} \in \bigcap_{k=0}^{2} H^k([0, T]; H^{4-2k}(\Omega)) \tag{29}$$

and we have

$$\|\mathbf{u}^{m+1}\|_{0,4} + \|\mathbf{u}^{m+1}\|_{1,2} + \|\mathbf{u}^{m+1}\|_{2,0} \leq \phi_1(M, T, K), \tag{30}$$

where $\phi_1(M, T, K)$ may depend on the initial value $\mathbf{u}_0$ and $\mathbf{u}_1$, and it is bounded if the parameters M, T and K are bounded.

Lemma 3. *Consider* (16) *and a second equation*

$$\mathbf{v}_t^{m+1} + (\mathbf{v}^m \cdot \nabla)\mathbf{v}^{m+1} + \nabla q^{m+1} = \Delta \mathbf{v}^{m+1} + \nabla \cdot \pi^m, \quad \nabla \cdot \mathbf{v}^{m+1} = 0 \tag{31}$$

and the initial and boundary conditions are

$$\mathbf{v}^{m+1}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}); \quad \mathbf{v}^{m+1} = 0, \quad on\ \Gamma. \tag{32}$$

Here we assume $\mathbf{v}^m \in S(M, T)$, $\mathbf{v}^m = \mathbf{u}^m$, $\pi^m = \tau^m$ for $t = 0$, and the assumptions of Lemma 2 *also hold for* (31) (*with the same constants M, K*). *Then we have an estimate of the form*

$$\begin{aligned}
&\|\mathbf{u}^{m+1} - \mathbf{v}^{m+1}\|_{0,4} + \|\mathbf{u}^{m+1} - \mathbf{v}^{m+1}\|_{1,2} + \|\mathbf{u}^{m+1} - \mathbf{v}^{m+1}\|_{2,0} \\
&\quad \leq \phi_2(M, T, K) \cdot (\|\mathbf{u}^m - \mathbf{v}^m\|_{0,4} + \|\mathbf{u}^m - \mathbf{v}^m\|_{1,2} + \|\mathbf{u}^m - \mathbf{v}^m\|_{2,0} \\
&\qquad + \|\tau^m - \pi^m\|_{0,3} + \|\tau^m - \pi^m\|_{1,1}),
\end{aligned}$$

where $\phi_2(M, T, K)$ is similar as $\phi_1(M, T, K)$ in Lemma 2, *moreover* $\lim_{T\to 0} \phi_2(M, T, K) = 0$.

Lemma 4. *Given $\mathbf{u}^m \in S(M, T)$, there exists a unique solution of* (19)–(20) *which has the regularity*

$$\psi^{m+1} \in \bigcap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathscr{X}_0)). \tag{33}$$

Let $\|\cdot\|_{(n)}$ *be the norm in*

$$\bigcap_{k=0}^{1} H^k([0,T]; H^{3-2k}(\Omega, \mathscr{X}_{n,0})). \tag{34}$$

We have an estimate of the form

$$\|\psi^{m+1}\|_{(n)} \leq K_1(n, M, T), \tag{35}$$

where K_1 *is similar to* ϕ_1 *in Lemma* 2.

In fact, we obtain the regularity of ψ^{m+1} is better than (33) in the proof of Lemma 4. But we only need the estimate of ψ^{m+1} in the norm $\|\cdot\|_{(n)}$, so we only emphasize the estimate of (33).

Let us consider a second equation of the same form:

$$\hat{\psi}_t^{m+1} + (\mathbf{v}^m \cdot \nabla)\hat{\psi}^{m+1} = -\nabla_{\mathbf{Q}} \cdot [(\hat{\kappa}^m \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\hat{\psi}^{m+1}] + \triangle_{\mathbf{Q}}\hat{\psi}^{m+1}, \tag{36}$$

$$\hat{\psi}^{m+1}(\mathbf{x}, \mathbf{Q}, 0) = \hat{\psi}_0(\mathbf{x}, \mathbf{Q}). \tag{37}$$

Here $\hat{\kappa}^m = (\nabla \mathbf{v}^m)^T$ and we assume $\mathbf{v}^m|_{t=0} = \mathbf{u}_0$. In addition to (19), the following result holds.

Lemma 5. *Let* $\mathbf{u}^m, \mathbf{v}^m \in S(M, T)$ *be given. Then for every* n*, we have an estimate of the form*

$$\|\psi^{m+1} - \hat{\psi}^{m+1}\|_{(n)} \leq K_2(n, M, T)d(\mathbf{u}^m, \mathbf{v}^m), \tag{38}$$

where K_2 *is similar to* ϕ_2 *in Lemma* 3.

Lemma 6.

$$\tau^m \in \bigcap_{k=0}^{1} H^k([0,T]; H^{3-2k}(\Omega)) \tag{39}$$

and

$$\|\tau^m - \pi^m\|_{0,3} + \|\tau^m - \pi^m\|_{1,1} \leq K_2(\sigma + 2, M, T)d(\mathbf{u}^{m-1}, \mathbf{v}^{m-1}).$$

provided that $\psi^m, \hat{\psi}^m \in \bigcap_{k=0}^{1} H^k([0,T]; H^{3-2k}(\Omega, \mathscr{X}_0))$ *and* $\pi^m = \int_{\mathbb{R}^3} \mathbf{g}(\mathbf{Q}) \otimes \mathbf{Q}\hat{\psi}^m \, d\mathbf{Q}$. *Here* σ *is defined in the condition* **(G)**.

By combining Lemmas 2–6, it follows easily that $\mathscr{M}$ is a contraction in $S(M, T)$ if M is chosen sufficiently large and T is chosen sufficiently small, then Theorem 1 follows immediately. Next we will be concerned with the proofs of lemmas.

3. PROOF OF LEMMAS

3.1. Estimates of u

In this subsection we will give the proof of Lemmas 2 and 3.

Let $\mathbf{u}^m = v$, $\mathbf{u}^{m+1} = w$ and $q = p^{m+1}$. (16) can be rewritten as

$$w_t + (v \cdot \nabla)w + \nabla q = \triangle w + f, \quad \nabla \cdot w = 0, \tag{40}$$

$$w = 0 \text{ on } \Gamma, \quad w(x, 0) = u_0. \tag{41}$$

Here $f = \nabla \cdot \tau^m$. Next setting $z = w_t$, then (40) will be transformed to

$$z = \triangle w - (v \cdot \nabla)w - \nabla q + f, \quad \nabla \cdot w = 0. \tag{42}$$

After utilizing the operator $\partial/\partial t$ to (42), we obtain

$$z_t = \triangle z - (v \cdot \nabla)z - \nabla\gamma + h(w, \nabla w, f_t) \tag{43}$$

$$\nabla \cdot z = 0, \quad z = 0 \quad \text{on } \Gamma, \tag{44}$$

$$z(x, 0) = z_0(x) = \mathbf{u}_1. \tag{45}$$

Here $h(w, \nabla w, f_t) = -v_t \cdot \nabla w + f_t$ and $\gamma = q_t$.

If we know z, we can solve the Stokes' problem (42) to obtain w. Meanwhile, we will obtain the solution z by solving the parabolic problem (43)–(45) when w is given. Next we will solve these two problems by using the Galerkin approximation.

Let V be the space of all divergence-free vector in $H_0^1(\Omega)$, and let $\{\phi^i | i \in \mathbb{N}\}$ be a basis for V. We seek an approximation to z of the form

$$z^N(x, t) = \sum_{n=1}^{N} \alpha^n(t)\phi^n(x). \tag{46}$$

For given z^N, let w^N be the corresponding approximate solution of (42). Now we solve the following approximate version of (43). For $n = 1, 2, \ldots, N$, we require that

$$\int_\Omega \phi^n \left\{z_t^N + (v \cdot \nabla)z^N - \triangle z^N - P^N(h(w^n, \triangle w^n, f_t))\right\} dx = 0 \tag{47}$$

and we impose the initial conditions

$$z^N(x, 0) = P^N z_0(x). \tag{48}$$

Here P^N is the orthogonal projection in V onto the span of $\phi^1, \phi^2, \ldots, \phi^N$. Equations (47) and (48) is an initial value problem for a linear system of first-order ODE's, and existence and uniqueness of the solutions are trivial.

To obtain an estimate uniform in N, we replace ϕ^n by z^N. It is a standard procedure to obtain the following estimate (Temam, 1995)

$$\|z^N(t)\|_{L^2}^2 + \int_0^T \|\nabla z^N(s)\|_{L^2}^2\, ds \le \|P^N z_0\|_{L^2}^2 + C\int_0^T \|h\|_{L^2}^2\, ds. \tag{49}$$

This implies that

$$z^N \in L^\infty([0,T];L^2(\Omega)) \cap L^2([0,T];V) \tag{50}$$

if $h \in L^2([0,T];L^2(\Omega))$, which will be showed later. Now we replace ϕ_N by z_t^N to obtain

$$\int_0^T \|z_t^N\|_{L^2}^2\,ds + \|\nabla z^N\|_{L^2}^2 \le C\int_0^T \|h\|_{L^2}^2 + \|\nabla z_0^N\|^2, \tag{51}$$

which implies

$$z^N \in L^\infty([0,T];V), \quad z_t^N \in L^2([0,T];L^2(\Omega)) \tag{52}$$

if $h \in L^2([0,T];L^2(\Omega))$. From (50) and (52), it is obtained that the regularity through z^N of Eq. (43) (Evans, 1998)

$$\|z^N\|_{H^2(\Omega)}^2 \le C\big(\|z_t^N\|_{L^2(\Omega)}^2 + \|h\|_{L^2(\Omega)}^2 + \|z^N\|_{L^2(\Omega)}^2\big). \tag{53}$$

Therefore,

$$z^N \in L^\infty([0,T];V) \cap L^2([0,T];H^2(\Omega)) \tag{54}$$

provided that $h \in L^2([0,T];L^2(\Omega))$.

In order to show that $h \in L^2([0,T];L^2(\Omega))$, the same Galerkin technique is applied to w's Eq. (40). w^N satisfies

$$w_t^N = \Delta w^N - (v\cdot\nabla)w^N + P^N(f - \nabla q), \quad \nabla\cdot w^N = 0, \tag{55}$$

$$w^N = 0 \text{ on } \Gamma, \quad w^N(x,0) = \mathbf{u}_0. \tag{56}$$

Similar as the estimate (53), we have

$$\|w^N\|_{H^2(\Omega)}^2 \le C\big(\|w_t^N\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|w^N\|_{L^2(\Omega)}^2\big). \tag{57}$$

provided that $f \in L^2([0,T],L^2(\Omega))$ and $\mathbf{u}_0 \in H^2(\Omega)$, which implies that $h \in L^2([0,T];L^2(\Omega))$ when $f \in H^1([0,T],L^2(\Omega))$.

Higher regularity of w^N can be obtained similarly. Recalling the definition of f and the condition on τ, we have $f \in L^2([0,T];H^2(\Omega)) \cap H^1([0,T];L^2(\Omega))$, and correspondingly we obtain

$$w^N \in \bigcap_{k=0}^{2} H^k([0,T];H^{4-2k}(\Omega)) \cap W^{1,\infty}([0,T];H^1(\Omega)). \tag{58}$$

Passing $N \to \infty$, we complete the proof of Lemma 2.

Proof of Lemma 3 is based on the same type of estimates. Let $\mathbf{U}^{m+1} = \mathbf{u}^{m+1} - \mathbf{v}^{m+1}$, $s^{m+1} = p^{m+1} - q^{m+1}$, $\varrho^m = \tau^m - \pi^m$. Then $\mathbf{U}^{m+1}$ satisfies

$$\mathbf{U}_t^{m+1} + (\mathbf{v}^m\cdot\nabla)\mathbf{U}^{m+1} + \nabla s^{m+1} = \triangle\mathbf{U}^{m+1} + \nabla\cdot\varrho^m + \mathbf{U}^m\cdot\nabla\mathbf{v}^{m+1}.$$

Then along the argument of Lemma 2 with slight modification, we can obtain Lemma 3 by studying the equation of $\mathbf{U}^{m+1}$ since $\mathbf{v}^{m+1}$ satisfies the result of Lemma 2.

3.2. Estimates of τ

Proof of Lemma 6. From the definition of τ, it is straightforward to obtain the estimate of it from the assumption of ψ. It is easy to check that

$$\|\tau\|^2_{L^2([0,T],L^2(\Omega))} \leq C\int_0^T \int_\Omega \left(\int_{\mathbb{R}^3} |\mathbf{Q}|^2 f(\mathbf{Q})\psi d\mathbf{Q}\right)^2 d\mathbf{x}\, dt.$$

From the condition **(G)** and $\psi \in L^2([0,T], L^2(\Omega, \mathscr{X}_0))$ we obtain the estimate of τ in $L^2([0,T], L^2(\Omega))$. By Lemma 5 and 6 we can obtain

$$\begin{aligned}\|\tau^m - \pi^m\|_{L^2([0,T],\Omega)} &\leq C\|\psi^m - \hat{\psi}^m\|_{(\sigma+2)} \\ &\leq K_2(\sigma+2, M, T)d(\mathbf{u}^{m-1}, \mathbf{v}^{m-1}).\end{aligned}$$

Higher derivatives can be obtained similarly.

3.3. Estimates of ψ with Respect to Lagrangian Variables

We will obtain the estimate of ψ with respect to Lagrangian variables first, then translate them to the Eulerian variables. Consider the flow map

$$\frac{\partial}{\partial t}\mathbf{x}(\alpha, t) = \mathbf{u}^m(\mathbf{x}(\alpha, t), t), \quad \mathbf{x}(\alpha, 0) = \alpha, \tag{59}$$

where α denotes the Lagrangian coordinates, and define $\phi(\alpha, \mathbf{Q}, t) = \psi^{m+1}(\mathbf{x}(\alpha, t), \mathbf{Q}, t)$, then (5) can be rewritten in the form

$$\frac{\partial}{\partial t}\phi(\alpha, \mathbf{Q}, t) = -\nabla_{\mathbf{Q}} \cdot [(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi] + \triangle_{\mathbf{Q}}\phi \tag{60}$$

$$\phi(\alpha, \mathbf{Q}, 0) = \psi_0(\alpha, \mathbf{Q}). \tag{61}$$

In this subsection we still denote $(\nabla \mathbf{u}(\mathbf{x}(\alpha, t), t))^T$ by κ with the loss of confusion. It follows from the maximum principle that positivity is preserved, and by integrating both sides of (60) we find that

$$\int_{\mathbb{R}^3} \phi(\alpha, \mathbf{Q}, t) d\mathbf{Q} = \int_{\mathbb{R}^3} \phi(\alpha, \mathbf{Q}, 0) d\mathbf{Q} = 1 \quad \text{for all } t.$$

Because the coefficients of Eq. (60) are unbounded, standard existence results for parabolic equations cannot be used. Now we are strongly motivated by the work of Renardy (1991) to use a sequence of approximating problems with bounded coefficients, for which we derive uniform estimates.

Let $\chi(\mathbf{Q})$ be a C^∞-function such that $\chi(0) = 1$, χ is a monotone decreasing function of $|\mathbf{Q}|$, and $\chi(\mathbf{Q}) = |\mathbf{Q}|^{-\beta}$ for large $|\mathbf{Q}|$, where β is a sufficiently large number. For $N \in \mathbb{N}$, let $\chi_N(\mathbf{Q}) = \chi(\mathbf{Q}/N)$. We now consider (60) the approximate problem

$$\frac{\partial}{\partial t}\phi_N(\alpha, \mathbf{Q}, t) = -\nabla_{\mathbf{Q}} \cdot [\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_N] + \triangle_{\mathbf{Q}}\phi_N, \tag{62}$$

$$\phi_N(\alpha, \mathbf{Q}, 0) = \psi_0(\alpha, \mathbf{Q}). \tag{63}$$

The existence of solutions of the Cauchy problem (62) and (63) is standard results since the coefficients of (62) are bounded. Next we will consider the estimates ϕ_N and its derivatives with respect to α and $\mathbf{Q}$. In order for the convenience of the notation, we will denote

$$\phi_N^{(i_1,i_2,\ldots,i_m;j_1,j_2,\ldots,j_n)} = \partial^m_{Q_{i_1},Q_{i_2},\ldots,Q_{i_m}} \partial^n_{\alpha_{j_1},\alpha_{j_2},\ldots,\alpha_{j_n}} \phi_N, \tag{64}$$

and specially we define

$$\phi_N^{(i_1,i_2,\ldots,i_m;0)} = \partial^m_{Q_{i_1},Q_{i_2},\ldots,Q_{i_m}} \phi_N, \quad \phi_N^{(0;j_1,j_2,\ldots,j_n)} = \partial^n_{\alpha_{j_1},\alpha_{j_2},\ldots,\alpha_{j_n}} \phi_N. \tag{65}$$

The summation convention is assumed in the following content.

3.3.1. The Estimate of ϕ_N

Multiplying (62) with $|\mathbf{Q}|^{2n}$ and integrating by parts yields

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n}\phi_N\, d\mathbf{Q} = (4n^2+2n)\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n-2}\phi_N\, d\mathbf{Q} + 2n\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n-2}\chi_N(\mathbf{Q}) \\ \times \{\mathbf{Q}\cdot\kappa\cdot\mathbf{Q} - \mathbf{Q}\cdot\mathbf{g}(\mathbf{Q})\}\phi_N\, d\mathbf{Q}. \end{aligned} \tag{66}$$

Pay attention that $\mathbf{Q}\cdot\mathbf{g}(\mathbf{Q})$ is nonnegative thus we obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n}\phi_N\, d\mathbf{Q} \le (4n^2+2n)\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n-2}\phi_N\, d\mathbf{Q} + 2n|\kappa|\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n}\phi_N\, d\mathbf{Q}. \tag{67}$$

Application of the Gronwall's inequality and $\mathbf{u} \in S(M, T)$ yields

$$\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n}\phi_N\, d\mathbf{Q} \le K(n, M, T)\int_{\mathbb{R}^3} (1+|\mathbf{Q}|^{2n})\psi_0\, d\mathbf{Q}, \tag{68}$$

where $K(n, M, T)$ is bounded if n, M, T is bounded, so we have

$$\phi_N \in L^\infty([0,T], L^\infty(\Omega, \mathscr{X}_0)) \quad \text{if } \psi_0 \in H^2(\Omega, \mathscr{X}_0). \tag{69}$$

3.3.2. The Estimate of $\nabla^m_{\mathbf{Q}}\phi_N$ $(m = 1, 2, 3)$

Differentiating (62) with respect to Q_i, we obtain the following equation for $\phi_N^{(i;0)}$:

$$\frac{\partial}{\partial t}\phi_N^{(i;0)} = -\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_N^{(i;0)}\right] + \triangle_{\mathbf{Q}}\phi_N^{(i;0)} + \Theta_i \tag{70}$$

$$\phi_N^{(i;0)}(\alpha, \mathbf{Q}, 0) = \partial_{Q_i}\psi_0(\alpha, \mathbf{Q}), \tag{71}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where

$$\Theta_i = -\{\partial_{Q_i}\chi_N[\kappa_{jk}\cdot Q_k - g_j(\mathbf{Q})] + \chi_N(\mathbf{Q})[\kappa_{ji} - \partial_{Q_i}g_j(\mathbf{Q})]\}\phi_N^{(j;0)} \\ -\nabla_{\mathbf{Q}}\cdot\{\partial_{Q_i}[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\mathbf{g}(\mathbf{Q}))]\}\phi_N \\ = \Theta_i^+ + \Theta_i^-. \tag{72}$$

Here $\Theta_i^+ = \Theta_i \vee 0$ and $\Theta_i^- = -(\Theta_i \wedge 0)$.

Now we decompose $\phi_N^{(i;0)} = \phi_{N+}^{(i;0)} - \phi_{N-}^{(i;0)}$, in which $\phi_{N+}^{(i;0)}$ and $\phi_{N-}^{(i;0)}$ are solutions of the following problems,

$$\frac{\partial}{\partial t}\phi_{N+}^{(i;0)} = -\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\mathbf{g}(\mathbf{Q}))\phi_{N+}^{(i;0)}\right] + \triangle_{\mathbf{Q}}\phi_{N+}^{(i;0)} + \Theta_i^+ \tag{73}$$

$$\phi_{N+}^{(i;0)}(\alpha,\mathbf{Q},0) = \max\{\partial_{Q_i}\psi_0(\alpha,\mathbf{Q}),0\}, \tag{74}$$

and respectively,

$$\frac{\partial}{\partial t}\phi_{N-}^{(i;0)} = -\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\mathbf{g}(\mathbf{Q}))\phi_{N-}^{(i;0)}\right] + \triangle_{\mathbf{Q}}\phi_{N-}^{(i;0)} + \Theta_i^- \tag{75}$$

$$\phi_{N-}^{(i;0)}(\alpha,\mathbf{Q},0) = \max\{-\partial_{Q_i}\psi_0(\alpha,\mathbf{Q}),0\}. \tag{76}$$

Since $\phi_{N+}^{(i;0)}$ and $\phi_{N-}^{(i;0)}$ are both positive, we can now proceed as the estimate of ϕ_N. We obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3}|\mathbf{Q}|^{2n}\phi_{N+}^{(i;0)}\,d\mathbf{Q} = (4n^2+2n)\int_{\mathbb{R}^3}|\mathbf{Q}|^{2n-2}\phi_{N+}^{(i;0)}\,d\mathbf{Q} \\ + 2n\int_{\mathbb{R}^3}|\mathbf{Q}|^{2n-2}\chi_N(\mathbf{Q})\{\mathbf{Q}\cdot\kappa\cdot\mathbf{Q}-\mathbf{Q}\cdot\mathbf{g}(\mathbf{Q})\}\phi_{N+}^{(i;0)}\,d\mathbf{Q} \\ + \int_{\mathbb{R}^3}|\mathbf{Q}|^{2n}\Theta_i^+\,d\mathbf{Q}. \tag{77}$$

By using (72), we have

$$\int_{\mathbb{R}^3}|\mathbf{Q}|^{2n}\Theta_i^+\,d\mathbf{Q} \le \int_{\mathbb{R}^3}|\mathbf{Q}|^{2n}\phi_N|\nabla_{\mathbf{Q}}\cdot\{\partial_{Q_i}[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\mathbf{g}(\mathbf{Q}))]\}|\,d\mathbf{Q} \\ + \int_{\mathbb{R}^3}|\mathbf{Q}|^{2n}\left(\phi_{N+}^{(j;0)}+\phi_{N-}^{(j;0)}\right)\{|\partial_{Q_i}\chi_N|(|\kappa_{jk}||Q_k|+|g_j(\mathbf{Q})|) \\ + \chi_N(\mathbf{Q})(|\kappa_{ji}|+|\partial_{Q_i}g_j(\mathbf{Q})|)\}\,d\mathbf{Q} \tag{78}$$

The first integral on the right hand side of (78) involves only ϕ_N and no derivatives of ϕ_N; it can be controlled using (68). The integrand in the second integral can be estimated for large $|\mathbf{Q}|$ by a constant times $\chi_N(\mathbf{Q})(M|\mathbf{Q}|^{2n} + |\mathbf{Q}|^{2n-1}|\mathbf{g}(\mathbf{Q})|)$ $\sum_j(\phi_{N+}^{(j;0)} + \phi_{N-}^{(j;0)})$, Thus we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^3} |\mathbf{Q}|^{2n}\phi_{N+}^{(i;0)}\, d\mathbf{Q} \\
&\quad \le C_1(n, M)\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\phi_{N+}^{(i;0)}\, d\mathbf{Q} + C_2(n, M) \\
&\qquad - 2n\int |\mathbf{Q}|^{2n-2}\chi_N\mathbf{Q}\cdot\mathbf{g}(\mathbf{Q})\phi_{N+}^{(i;0)}\, d\mathbf{Q} + K_3\int_{\mathbb{R}^3}\{1 + |\mathbf{Q}|^{2n-1}\chi_N|\mathbf{g}(\mathbf{Q})| \\
&\qquad + M\chi_N|\mathbf{Q}|^{2n}\}\sum_j(\phi_{N+}^{(j;0)} + \phi_{N-}^{(j;0)})d\mathbf{Q}.
\end{aligned} \tag{79}$$

Similar estimate can be done for $\phi_{N-}^{(i;0)}$. Summing up all the index i and the $+, -$ equations we obtain an inequality of quantity $\int_{\mathbb{R}^3}|\mathbf{Q}|^{2n}\sum_i|\phi_N^{(i;0)}|d\mathbf{Q}$. Noticing when n is sufficiently large, the third integral may be controlled by the second integral, thus we obtain

$$\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_i|\phi_N^{(i;0)}|d\mathbf{Q} \le K(n, M, T)\Big(\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})(|\nabla_{\mathbf{Q}}\psi_0| + |\psi_0|)d\mathbf{Q}\Big). \tag{80}$$

It implies

$$\phi_N \in L^\infty\big([0, T]; L^\infty\big(\Omega, \mathscr{X}_1\big)\big) \quad \text{if } \psi_0 \in H^2\big(\Omega, \mathscr{X}_1\big). \tag{81}$$

By the same way, we may obtain for $k = 2$

$$\phi_N \in L^\infty\big([0, T]; L^\infty\big(\Omega, \mathscr{X}_2\big)\big) \quad \text{if } \psi_0 \in H^2(\Omega, \mathscr{X}_2). \tag{82}$$

3.3.3. The Estimate of $(\phi_N)_t$

We take the absolute value to the two sides of the symbol "=" to (62) and multiply it by $1 + |\mathbf{Q}|^{2n}$ and integrate it in $\mathbf{Q}$ space. It will be obtained that

$$(\phi_N)_t \in L^\infty\big([0, T]; L^\infty\big(\Omega, \mathscr{X}_0\big)\big) \quad \text{if } \psi_0 \in H^2\big(\Omega, \mathscr{X}_2\big). \tag{83}$$

3.3.4. The Estimate of $\nabla_\alpha\phi_N$

In the following we will show that $\int_{\mathbb{R}^3}|\nabla_\alpha\phi_N|d\mathbf{Q}$ is bounded. Differentiating (62) with respect to α_i yields the equation

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\phi_N^{(0;i)} = -\nabla_{\mathbf{Q}}\cdot\big[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_N^{(0;i)}\big] + \triangle_{\mathbf{Q}}\phi_N^{(0;i)} + \Psi_i \\
&\phi_N^{(0;i)}(\alpha, \mathbf{Q}, 0) = \partial_{\alpha_i}\psi_0(\alpha, \mathbf{Q}),
\end{aligned} \tag{84}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where

$$\begin{aligned}\Psi_i &= -\nabla_{\mathbf{Q}} \cdot [\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa \cdot \mathbf{Q}\phi_N] \\ &= -\nabla_{\mathbf{Q}} \cdot [\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa \cdot \mathbf{Q}]\phi_N - [\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa \cdot \mathbf{Q}] \cdot \nabla_{\mathbf{Q}}\phi_N \\ &= \Psi_i^+ + \Psi_i^-. \end{aligned} \tag{85}$$

Here Ψ_i^+ and Ψ_i^- have the similar definition as before.

Let $\phi_N^{(0;i)} = \phi_{N+}^{(0;i)} - \phi_{N-}^{(0;i)}$, where $\phi_{N+}^{(0;i)}$ and $\phi_{N-}^{(0;i)}$ are the solutions of the following problems

$$\frac{\partial}{\partial t}\phi_{N\pm}^{(0;i)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_{N\pm}^{(0;i)}\right] + \triangle_{\mathbf{Q}}\phi_{N\pm}^{(0;i)} + \Psi_i^\pm \tag{86}$$

$$\phi_{N\pm}^{(0;i)}(\alpha, \mathbf{Q}, 0) = (\partial_{\alpha_i}\psi_0)^\pm. \tag{87}$$

Because $\phi_{N+}^{(0;i)}$ and $\phi_{N-}^{(0;i)}$ are positive, we can proceed as above once more. Observing that there are only ϕ_N and $\phi_N^{(j;0)}$ terms involved in Ψ_i, and using $\nabla_\alpha\kappa \in L^2([0, T], L^\infty(\Omega))$ (see formula (119)) we will obtain

$$\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_i |\phi_N^{(0;i)}|d\mathbf{Q} \le K(n, M, T)\Big(\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})(|\nabla_\alpha\psi_0| + |\psi_0|)d\mathbf{Q}\Big). \tag{88}$$

That shows

$$\nabla_\alpha\phi_N \in L^\infty([0, T]; L^\infty(\Omega, \mathscr{X}_0)) \quad \text{if } \psi_0 \in H^3(\Omega, \mathscr{X}_2). \tag{89}$$

3.3.5. The Estimate of $\nabla_{\mathbf{Q}}\nabla_\alpha\phi_N$ and $\nabla_{\mathbf{Q}}^2\nabla_\alpha\phi_N$

Next we will estimate the mixed derivatives of ϕ_N. Differentiating (84) with respect to Q_j yields the equation of $\phi_N^{(j;i)}$,

$$\frac{\partial}{\partial t}\phi_N^{(j;i)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_N^{(j;i)}\right] + \triangle_{\mathbf{Q}}\phi_N^{(j;i)} + \Lambda_{ji} \tag{90}$$

$$\phi_N^{(j;i)}(\alpha, \mathbf{Q}, 0) = \partial_{Q_j}\partial_{\alpha_i}\psi_0(\alpha, \mathbf{Q}), \tag{91}$$

where

$$\begin{aligned}\Lambda_{ji} = &-\{\partial_{Q_j}\chi_N[\kappa_{lk} \cdot Q_k - g_l(\mathbf{Q})] + \chi_N(\mathbf{Q})[\kappa_{lj} - \partial_{Q_j}g_l(\mathbf{Q})]\}\phi_N^{(l;i)} \\ &-\nabla_{\mathbf{Q}} \cdot \{\partial_{Q_j}[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))]\}\phi_N^{(0;i)} \\ &-\nabla_{\mathbf{Q}} \cdot [\partial_{Q_j}\chi_N\partial_{\alpha_i}\kappa \cdot \mathbf{Q}\phi_N] - \partial_{Q_l}[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa_{lj}\phi_N] \\ &-\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa \cdot \mathbf{Q}\phi_N^{(j;0)}\right] \\ = &\ \Lambda_{ji}^+ + \Lambda_{ji}^-. \end{aligned} \tag{92}$$

Here Λ_{ji}^+ and Λ_{ji}^- have the similar definition as before.

Let $\phi_N^{(j;i)} = \phi_{N+}^{(j;i)} - \phi_{N-}^{(j;i)}$, where $\phi_{N+}^{(j;i)}$ and $\phi_{N-}^{(j;i)}$ are the solutions of the following problems

$$\frac{\partial}{\partial t}\phi_{N\pm}^{(j;i)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_{N\pm}^{(j;i)}\right] + \triangle_{\mathbf{Q}}\phi_{N\pm}^{(j;i)} + \Lambda_{ji}^{\pm} \tag{93}$$

$$\phi_{N\pm}^{(j;i)}(\alpha, \mathbf{Q}, 0) = (\partial_{Q_j}\partial_{\alpha_i}\psi_0)^{\pm}. \tag{94}$$

Because $\phi_{N+}^{(j;i)}$ and $\phi_{N-}^{(j;i)}$ are positive, we can proceed as before. The terms involving $\phi_N^{(l;i)}$ in Λ_{ji} may be controlled by the terms from (90) by integrating by parts as in (79), and the other terms has been estimated in former subsections. Then by using $\nabla_\alpha\kappa$ and κ belong to $L^2([0,T], L^\infty(\Omega))$ we find

$$\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{ij}\left|\phi_N^{(j;i)}\right|d\mathbf{Q}$$
$$\leq K(n,M,T)\left(\int_{\mathbb{R}^3}\left(1+|\mathbf{Q}|^{2n}\right)\left[\sum_{l,m=0}^{1}|\nabla_{\mathbf{Q}}^{l}\nabla_{\alpha}^{m}\psi_0(\alpha,\mathbf{Q},t)|\right]d\mathbf{Q}\right). \tag{95}$$

That shows

$$\nabla_\alpha\phi_N \in L^\infty([0,T]; L^\infty(\Omega, \mathscr{X}_1)) \quad \text{if } \psi_0 \in H^3(\Omega, \mathscr{X}_2). \tag{96}$$

By the same way we can obtain that

$$\nabla_\alpha\phi_N \in L^\infty([0,T]; L^\infty(\Omega, \mathscr{X}_2)) \quad \text{if } \psi_0 \in H^3(\Omega, \mathscr{X}_2). \tag{97}$$

3.3.6. The Estimate of $(\nabla_\alpha\phi_N)_t$

We take absolute value to both sides of the symbol "=" to (84) and multiply it by $|\mathbf{Q}|^{2n}$ and integrate it to $\mathbf{Q}$. Moreover we have to integrate it to α since the factor $\partial_{\alpha_i}\kappa$ of Ψ_i in (84) is only belong to $L^2([0,T], L^\infty(\Omega))$. After the utilization of (96) and (97) it will be obtained that

$$(\nabla_\alpha\phi_N)_t \in L^2([0,T]; L^\infty(\Omega, \mathscr{X}_0)) \quad \text{if } \psi_0 \in H^3(\Omega, \mathscr{X}_2). \tag{98}$$

This yields

$$\phi_N \in H^1([0,T]; H^1(\Omega; \mathscr{X}_0)) \tag{99}$$

if $\psi_0 \in H^3(\Omega; \mathscr{X}_2)$.

3.3.7. The Estimate of $\nabla_\alpha^2 \phi_N$

Next we estimate the high order derivative of ϕ_N with respect to α. Differentiating (84) with respect to α_j yields the equation

$$\frac{\partial}{\partial t}\phi_N^{(0;i,j)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_N^{(0;i,j)}\right] + \triangle_{\mathbf{Q}}\phi_N^{(0;i,j)} + \Pi_{ij} \tag{100}$$

$$\phi_N^{(0;i,j)}(\alpha, \mathbf{Q}, 0) = \partial^2_{\alpha_i\alpha_j}\psi_0(\alpha, \mathbf{Q}). \tag{101}$$

where

$$\begin{aligned}\Pi_{ij} &= -\nabla_{\mathbf{Q}} \cdot [\chi_N(\mathbf{Q})\partial^2_{\alpha_i\alpha_j}\kappa \cdot \mathbf{Q}\phi_N] - \nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa \cdot \mathbf{Q}\phi_N^{(0;j)}\right]\\ &\quad -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})\partial_{\alpha_j}\kappa \cdot \mathbf{Q}\phi_N^{(0;i)}\right]\\ &= \Pi_{ij}^+ + \Pi_{ij}^-.\end{aligned} \tag{102}$$

Here Π_{ij}^+ and Π_{ij}^- have the similar definition as before.

As before we decompose $\phi_N^{(0;i,j)} = \phi_{N+}^{(0;i,j)} - \phi_{N-}^{(0;i,j)}$, where $\phi_{N+}^{(0;i,j)}$ and $\phi_{N-}^{(0;i,j)}$ satisfy

$$\frac{\partial}{\partial t}\phi_{N\pm}^{(0;i,j)} = -\nabla_{\mathbf{Q}} \cdot \left[\chi_N(\mathbf{Q})(\kappa \cdot \mathbf{Q} - \mathbf{g}(\mathbf{Q}))\phi_{N\pm}^{(0;i,j)}\right] + \triangle_{\mathbf{Q}}\phi_{N\pm}^{(0;i,j)} + \Pi_{ij}^\pm \tag{103}$$

$$\phi_{N\pm}^{(0;i,j)}(\alpha, \mathbf{Q}, 0) = (\partial^2_{\alpha_i\alpha_j}\psi_0)^\pm. \tag{104}$$

From (102) we have

$$\begin{aligned}&\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\Pi_{ij}^+ \, d\mathbf{Q}\\ &\quad\le |\nabla_\alpha^2\kappa|\left(\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\phi_N \, d\mathbf{Q} + \int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_k |\phi_N^{(k;0)}|d\mathbf{Q}\right)\\ &\qquad + |\nabla_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_l |\phi_N^{(0;l)}|d\mathbf{Q} + \int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_{k,l} |\phi_N^{(k;l)}|d\mathbf{Q}\right).\end{aligned} \tag{105}$$

Multiplying both sides of (103) by $(1 + |\mathbf{Q}|^{2n})$, integrating to $\mathbf{Q}$ and summing them altogether, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})|\phi_N^{(0;i,j)}|d\mathbf{Q}\\ &\quad\le K(M, n)\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})|\phi_N^{(0;i,j)}|d\mathbf{Q}\\ &\qquad + |\nabla_\alpha^2\kappa|\left(\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\phi_N \, d\mathbf{Q} + \int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_k |\phi_N^{(k;0)}|d\mathbf{Q}\right)\\ &\qquad + |\nabla_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_l |\phi_N^{(0;l)}|d\mathbf{Q} + \int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\sum_{k,l} |\phi_N^{(k;l)}|d\mathbf{Q}\right)\end{aligned} \tag{106}$$

Now we multiply $\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j)}|d\mathbf{Q}$ to (106) and integrate it in α space. It can be obtained

$$\frac{\partial}{\partial t}\int_\Omega\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j)}|d\mathbf{Q}\right)^2 d\alpha$$
$$\leq K(M,n)\int_\Omega\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j)}|d\mathbf{Q}\right)^2 d\alpha + C_1\int_\Omega|\nabla_\alpha^2\kappa|^2\,d\alpha + C_2\|\nabla_\alpha\kappa\|_{L^\infty}^2. \tag{107}$$

Here we used that

$$\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\phi_N\,d\mathbf{Q},\quad \int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_k|\phi_N^{(k;0)}|d\mathbf{Q}$$
$$\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(0;l)}|d\mathbf{Q},\quad \int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{k,l}|\phi_N^{(k;l)}|d\mathbf{Q}$$

belong to $L^\infty([0,T]\times\Omega)$ and the boundedness of Ω. By using $|\nabla_\alpha\kappa|\in L^2([0,T], L^\infty(\Omega))$ and Gronwall's inequality, we obtain

$$\phi_N\in L^\infty([0,T];H^2(\Omega,\mathscr{X}_0))\quad \text{if } \psi_0\in H^3(\Omega,\mathscr{X}_2). \tag{108}$$

3.3.8. The Estimate of $\nabla_\alpha^3\phi_N$

Differentiating (100) with respect to α yields the equation

$$\frac{\partial}{\partial t}\phi_N^{(0;i,j,k)} = -\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\mathbf{g}(\mathbf{Q}))\phi_N^{(0;i,j,k)}\right] + \triangle_{\mathbf{Q}}\phi_N^{(0;i,j,k)} + \Upsilon_{ijk} \tag{109}$$
$$\phi_N^{(0;i,j,k)}(\alpha,\mathbf{Q},0) = \partial^3_{\alpha_i\alpha_j\alpha_k}\psi_0(\alpha,\mathbf{Q}), \tag{110}$$

where

$$\begin{aligned}\Upsilon_{ijk} = &-\nabla_{\mathbf{Q}}\cdot[\chi_N(\mathbf{Q})\partial^3_{\alpha_i\alpha_j\alpha_k}\kappa\cdot\mathbf{Q}\phi_N] - \nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})\partial^2_{\alpha_i\alpha_j}\kappa\cdot\mathbf{Q}\phi_N^{(0;k)}\right]\\ &-\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})\partial^2_{\alpha_i\alpha_k}\kappa\cdot\mathbf{Q}\phi_N^{(0;j)}\right] - \nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})\partial^2_{\alpha_j\alpha_k}\kappa\cdot\mathbf{Q}\phi_N^{(0;i)}\right]\\ &-\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})\partial_{\alpha_i}\kappa\cdot\mathbf{Q}\phi_N^{(0;j,k)}\right] - \nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})\partial_{\alpha_j}\kappa\cdot\mathbf{Q}\phi_N^{(0;i,k)}\right]\\ &-\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})\partial_{\alpha_k}\kappa\cdot\mathbf{Q}\phi_N^{(0;i,j)}\right]\\ = &\ \Upsilon^+_{ijk}+\Upsilon^-_{ijk}.\end{aligned} \tag{111}$$

Here Υ^+_{ijk} and Υ^-_{ijk} have the similar definition as before.

As before we decompose $\phi_N^{(0;i,j,k)} = \phi_{N+}^{(0;i,j,k)} - \phi_{N-}^{(0;i,j,k)}$, where $\phi_{N+}^{(0;i,j,k)}$ and $\phi_{N-}^{(0;i,j,k)}$ are satisfied

$$\frac{\partial}{\partial t}\phi_{N\pm}^{(0;i,j,k)} = -\nabla_{\mathbf{Q}}\cdot\left[\chi_N(\mathbf{Q})(\kappa\cdot\mathbf{Q}-\mathbf{g}(\mathbf{Q}))\phi_{N\pm}^{(0;i,j,k)}\right] + \triangle_{\mathbf{Q}}\phi_{N\pm}^{(0;i,j,k)} + \Upsilon^\pm_{ijk} \tag{112}$$
$$\phi_{N\pm}^{(0;i,j,k)}(\alpha,\mathbf{Q},0) = \left(\partial^3_{\alpha_i\alpha_j\alpha_k}\psi_0\right)^\pm. \tag{113}$$

By (111) we can find that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\Upsilon^{+}_{ijk}\,d\mathbf{Q}\\
&\quad\le|\nabla^3_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\phi_N\,d\mathbf{Q}+\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(l;0)}|d\mathbf{Q}\right)\\
&\qquad+|\nabla^2_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(0;l)}|d\mathbf{Q}+\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{m,n}|\phi_N^{(m;n)}|d\mathbf{Q}\right)\\
&\qquad+|\nabla_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{m,n}|\phi_N^{(0;m,n)}|d\mathbf{Q}+\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{l,m,n}|\phi_N^{(l;m,n)}|d\mathbf{Q}\right).
\end{aligned}
\tag{114}
$$

Multiplying both sides of (112) by $(1+|\mathbf{Q}|^{2n})$, and integrating to $\mathbf{Q}$ and summing them altogether, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j,k)}|d\mathbf{Q}\\
&\quad\le K(M,n)\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j,k)}|d\mathbf{Q}\\
&\qquad\times|\nabla^3_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\phi_N\,d\mathbf{Q}+\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(l;0)}|d\mathbf{Q}\right)\\
&\qquad+|\nabla^2_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(0;l)}|d\mathbf{Q}+\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{m,n}|\phi_N^{(m;n)}|d\mathbf{Q}\right)\\
&\qquad+|\nabla_\alpha\kappa|\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{m,n}|\phi_N^{(0;m,n)}|d\mathbf{Q}+\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{l,m,n}|\phi_N^{(l;m,n)}|d\mathbf{Q}\right).
\end{aligned}
\tag{115}
$$

Now we multiply $\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j,k)}|d\mathbf{Q}$ to (115) and integrate it to α. It can be found

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j,k)}|d\mathbf{Q}\right)^2d\alpha\\
&\quad\le K(M,n)\int_\Omega\left(\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})|\phi_N^{(0;i,j,k)}|d\mathbf{Q}\right)^2d\alpha\\
&\qquad+C_1\int_\Omega(|\nabla^3_\alpha\kappa|^2+|\nabla^2_\alpha\kappa|^2)d\alpha+C_2\|\nabla_\alpha\kappa\|^2_{L^\infty}
\end{aligned}
\tag{116}
$$

Here we used that

$$
\Big|\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\phi_N\,d\mathbf{Q},\quad\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(0;l)}|d\mathbf{Q},
$$

$$
\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_l|\phi_N^{(l;0)}|d\mathbf{Q},\quad\int_{\mathbb{R}^3}(1+|\mathbf{Q}|^{2n})\sum_{m,n}|\phi_N^{(m;n)}|d\mathbf{Q}
$$

belong to $L^\infty([0,T]; H^2(\Omega, \mathscr{X}_0))$ and the boundedness of Ω. Application of Gronwall's inequality and $\nabla_\alpha \kappa \in L^2([0,T]; L^\infty(\Omega))$ implies that

$$\phi_N \in L^\infty([0,T]; H^3(\Omega, \mathscr{X}_0)) \quad \text{if } \psi_0 \in H^3(\Omega, \mathscr{X}_2). \tag{117}$$

3.4. Estimate of ψ with Respect to Eulerian Variable

3.4.1. The Estimate of Flow Map

Since we need the estimates of the derivatives of ψ to $\mathbf{x}$ but now we have only those of ψ to α, Eq. (59) implies that we will have them if we can obtain the estimate of $\mathbf{x}(\alpha, t)$ of (59).

Since $\mathbf{u}^m \in S(M,T)$, we have $\nabla \mathbf{u}^m \in L^\infty([0,T] \times \Omega)$ (see Evans, 1998, pp. 287). From Eq. (59) we have

$$\frac{\partial}{\partial t}(\nabla_\alpha \mathbf{x}) = \nabla \mathbf{u}^m \cdot \nabla_\alpha \mathbf{x}. \tag{118}$$

By utilizing this equation we know $\nabla_\alpha \mathbf{x} \in L^\infty([0,T] \times \Omega)$, thus it is deduced that

$$\nabla_\alpha \kappa = \nabla[(\nabla \mathbf{u}^m(\mathbf{x}(\alpha,t),t))^T] \cdot \nabla_\alpha \mathbf{x} \in L^2([0,T], L^\infty(\Omega)). \tag{119}$$

Moreover, $\partial/\partial t(\nabla\alpha \cdot \nabla_\alpha \mathbf{x}) = 0$ implies

$$\frac{\partial}{\partial t}\nabla\alpha = -\nabla\alpha \cdot \frac{\partial}{\partial t}(\nabla_\alpha \mathbf{x}) \cdot (\nabla_\alpha \mathbf{x})^{-1}. \tag{120}$$

Combining of (118) and (120) yields

$$\frac{\partial}{\partial t}(\nabla\alpha) = \nabla\alpha \cdot \nabla \mathbf{u}^m. \tag{121}$$

From $\nabla \mathbf{u}^m \in L^\infty([0,T] \times \Omega)$ we obtain

$$\nabla\alpha \in L^\infty([0,T] \times \Omega). \tag{122}$$

Now by using (121)–(122) we have

$$\frac{\partial}{\partial t}\|\nabla^2\alpha\|^2_{L^2(\Omega)} \le C_1\|\nabla^2\alpha\|^2_{L^2(\Omega)} + C_2\|\nabla^2\mathbf{u}^m\|^2_{L^2(\Omega)}. \tag{123}$$

The Gronwall's inequality implies

$$\nabla^2\alpha \in L^\infty([0,T]; L^2(\Omega)) \tag{124}$$

when $\mathbf{u}^m \in S(M,T)$. Similarly we can find

$$\frac{\partial}{\partial t}\|\nabla^3\alpha\|^2_{L^2(\Omega)} \le C_1\|\nabla^3\alpha\|^2_{L^2(\Omega)} + C_2\|\nabla^2\alpha\|^2_{L^2(\Omega)} + C_3\|\nabla^3\mathbf{u}^m\|^2_{L^2(\Omega)}. \tag{125}$$

It implies if $\mathbf{u}^m \in S(M,T)$,

$$\nabla^3\alpha \in L^\infty([0,T]; L^2(\Omega)). \tag{126}$$

3.4.2. Estimate of ϕ_N w.r.t. $\mathbf{x}$

By (122), (124), (126) and

$$\nabla\phi_N(\mathbf{x}, \mathbf{Q}, t) = \nabla_\alpha\phi_N \cdot \nabla\alpha \tag{127}$$

we have

$$\phi_N(\mathbf{x}, \mathbf{Q}, t) \in \bigcap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathscr{X}_0)) \tag{128}$$

when $\mathbf{u}^m \in S(M, T)$ and

$$\phi_N(\alpha, \mathbf{Q}, t) \in \bigcap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathscr{X}_0)). \tag{129}$$

3.4.3. The Limit of ϕ_N When $N \to \infty$

All of the estimates above are uniform in N and the restriction to the initial data is $\psi_0 \in \bigcap_{i=0}^{2} H^3(\Omega, \mathscr{X}_i)$. But when N passes to ∞, the limit of the sequence ϕ_N may not in $\bigcap_{k=0}^{1} H^k([0, T]; H^{3-2k}(\Omega, \mathscr{X}_0))$ since $\mathscr{X}_k$ are based in L^1-type norms and the limit of a distributionally convergent bounded sequence in L^1 may not be an L^1-function, but a singular measure. In order to overcome the difficulty, now we can utilize the technology in Renardy (1991) to improve the regularity of ψ_0. This is the cause to suppose $\psi_0 \in H^4(\Omega, \mathscr{X}_2)$ in the condition (**A3**) of Theorem 1. The detail is similar to that in Renardy (1991).

The proof of Lemma 5 is based on the same type of estimates, applied to the function $\psi - \hat{\psi}$. We omit the details.

4. CONCLUSION

A detailed well-posedness analysis for the dumbbell model of polymeric fluids is finished in this paper. The model considered is a coupled system of fluids velocity $\mathbf{u}$ and distribution density ψ for polymeric fluids in kinetic theory of polymers. The main theoretical framework is originally in the paper Renardy (1991), but a more detailed analysis is focused on the derivative estimate of ψ with respect to space variable α. In particular, the L^∞ norm of $\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})\phi_N\, d\mathbf{Q}$, $\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})|\nabla_\mathbf{Q}\phi_N| d\mathbf{Q}$, $\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})|\nabla_\mathbf{x}\phi_N| d\mathbf{Q}$ and $\int_{\mathbb{R}^3}(1 + |\mathbf{Q}|^{2n})|\nabla_\mathbf{x}\nabla_\mathbf{Q}\phi_N| d\mathbf{Q}$ are essential for the proof, which is the key point of the priori estimate. This work is crucial for the numerical analysis of the recently proposed multiscale methods for solving (1)–(5) (Jendrejak et al., 2002).

ACKNOWLEDGMENTS

The authors are very grateful to Professor Weinan E for his helpful discussions. Hui Zhang acknowledges the financial support of the Post-Doctoral Science Foundation of China. Pingwen Zhang is partially supported by the special funds for Major State Research Projects G1999032804 and National Science Foundation of China for Distinguished Young Scholars 10225103.

REFERENCES

Bird, R. B., Armstrong, R. C., Hassager, O. (1987). *Dynamics of Polymeric Liquids*. Vol. 1 and 2. 2nd ed. New York: John Wiley.

Doi, M., Edwards, S. F. (1986). *The Theory of Polymer Dynamics*. Oxford: Oxford University Press.

Evans, L. C. (1998). *Partial Differential Equations*. Providence, Rhode Island: AMS.

Feigl, K., Laso, M., Öttinger, H. C. (1995). CONNFFESSIT approach for solving a two-dimensional viscoelastic fluids problem. *Macromolecules* 28:3261–3274.

Guillopé, C., Saut, J. C. (1990). Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal. TMA* 15:849–869.

Hulsen, M. A., van Heel, A. P. G., van den Brule, B. H. A. A. (1997). Simulation of viscoelastic flows using Browmian configuration field. *J. Non-Newtonian Fluid Mech.* 70:79–101.

Jendrejak, R. M., de Pablo, J. J., Graham, M. D. (2002). A method for multiscale simulation of flowing complex fluid. *J. Non-Newtonian Fluid Mech.* 108: 123–142.

Jourdain, B., Lelievre, T., Le Bris, C. (2002). Numerical analysis of micro-macro simulations of polymeric fluid flows: A simple case. *Math. Models Methods Appl. Sci.* 12(9):1205–1243.

Jourdain, B., Lelievre, T., Le Bris, C. (2004). Existence of solution for a micro-macro model of polymeric fluid: The FENE model. *J. Funct. Anal.* 209: 162–193.

Laso, M., Öttinger, H. C. (1993). Calculation of viscoelastic flow using molecuar models: The CONNFFESSIT approach. *J. Non-Newtonian Fluid Mech.* 47:1–20.

Li, T. E. W., Zhang, P. (2002). Convergence of a stochastic method for the modeling of polymeric fluids. *Acta Mathematicae Applicatae Sinica, English Series* 18(4):529–536.

Li, T. E. W., Zhang, P. (2003). Well-Posedness for the dumbbell model of polymeric fluids. Accepted by *Comm. Math. Phys.*

Lions, P. L., Masmoudi, N. (2000). Global solutios for some Oldroyd models of non-Newtonian flows. *Chin. Ann. Math.* 21B:131–146.

Renardy, M. (1990). Local existence of solutions of the Dirichlet initial boundary problem for incompressible hypoelastic materials. *SIAM J. Math. Anal.* 21:1369–1385.

Renardy, M. (1991). An existence theorem for model equations resulting from kinetic theories. *SIAM J. Math. Anal.* 22(2):313–327.

Risken, H. (1984). *The Fokker-Planck Equation, Methods of Solution and Applications*. Springer Verlag.

Temam, R. (1995). *Navier-Stokes Equations and Nonlinear Functional Analysis*. Philadelphia: SIAM.

Received June 2003
Revised December 2003