MULTISCALE MODEL. SIMUL.
Vol. 5, No. 1, pp. 205–234

CONVERGENCE ANALYSIS OF BCF METHOD FOR HOOKEAN DUMBBELL MODEL WITH FINITE DIFFERENCE SCHEME*

TIEJUN LI† AND PINGWEN ZHANG‡

Abstract. A convergence analysis of the Brownian configuration fields (BCF) method [M. A. Hulsen, A. P. G. van Heel, and B. H. A. A. van den Brule, *J. Non-Newtonian Fluid Mech.*, 70 (1997), pp. 79–101] for the Hookean dumbbell model with finite difference scheme in dimension 2 or 3 is given in this paper under the assumption that the continuous solution is smooth enough. An explicit solution of the Hookean dumbbell model is obtained via deformation tensor. A large deviation-type estimate for the error of polymeric stress $\mathbb{E}(\boldsymbol{QQ})$ is given, which is a key step in the proof. It is shown that if the number of configuration fields N, the space stepsize h, and the time stepsize δt are chosen appropriately, the convergence of second order in space and first order in time may be proved after excluding a set of small probability. Simultaneous discretization of Monte Carlo and space and the inverse inequality trick are essential for the proof.

Key words. Brownian configuration fields method, Hookean dumbbell model, convergence analysis, large deviation

AMS subject classifications. 65C30, 76M35, 82D60

DOI. 10.1137/05063430X

1. Introduction.

The dumbbell model is the simplest model of polymeric fluids that takes into account the microscopic behavior of the solute polymers [2, 4, 16]. It models the polymers in dilute solution by dumbbells, each with two beads connected by a spring. The configuration of the spring then specifies the conformation of the polymer. Denote by $\boldsymbol{u}$ and p the velocity and pressure of the fluid and by $\boldsymbol{Q}$ the configuration of the spring; then one has the following coupled macroscopic-kinetic equations:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} + \nabla\cdot\mathbb{E}(\boldsymbol{F}(\boldsymbol{Q})\boldsymbol{Q}), \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1.2}$$

$$d\boldsymbol{Q} = (-\boldsymbol{u}\cdot\nabla\boldsymbol{Q} + \kappa\boldsymbol{Q} - \boldsymbol{F}(\boldsymbol{Q}))dt + d\boldsymbol{W}_t, \tag{1.3}$$

where $\boldsymbol{F}(\boldsymbol{Q})$ is the spring force, $\kappa = \nabla\boldsymbol{u}^T$ is the strain rate, $\boldsymbol{W}_t$ is the standard multidimensional Wiener process only in time, $\boldsymbol{F}(\boldsymbol{Q})\boldsymbol{Q}$ is understood as tensor product, and $\mathbb{E}$ is the expectation of random variables. Here all of the physical constants are taken to be 1 for simplicity. In general, $\boldsymbol{F}(\boldsymbol{Q}) = \gamma(|Q|^2)\boldsymbol{Q}$, and $\gamma(|Q|^2) \geq 0$. The corresponding initial and boundary conditions are supplied for different problems. In the case of $\boldsymbol{F}(\boldsymbol{Q}) = \boldsymbol{Q}$, the Hookean dumbbell model, the polymeric stress $\tau = \mathbb{E}(\boldsymbol{QQ})$

*Received by the editors June 23, 2005; accepted for publication (in revised form) December 16, 2005; published electronically April 21, 2006.
http://www.siam.org/journals/mms/5-1/63430.html
†Corresponding author. LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China (tieli@pku.edu.cn). The research of this author was supported by National Science Foundation of China grants 10401004 and 20490222 and the National Basic Research Program under grant 2005CB321704.
‡LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China (pzhang@pku.edu.cn). The research of this author was supported by National Science Foundation of China grant 20490222, the National Basic Research Program under grant 2005CB321704, and Distinguished Young Scholars grant 10225103.

satisfies the well-known Oldroyd-B model

$$\tau + \overset{\nabla}{\tau} = \boldsymbol{I}, \tag{1.4}$$

where $\boldsymbol{I}$ is the identity matrix and

$$\overset{\nabla}{\tau} := \partial_t \tau + (\boldsymbol{u} \cdot \nabla)\tau - \kappa \cdot \tau - \tau \cdot \kappa^T \tag{1.5}$$

is the upper convective derivative [1]. In a general case one would not expect a closure equation for the stress τ.

The system (1.1), (1.2), and (1.3) has its meaning in the multiscale simulation of complex fluids. Usually a flexible polymer in the solvent is modelled as a connected elastic dumbbell. In the dilute case, the probability density function f satisfies the well-known Fokker–Planck equations which lie in a very high dimensional configuration space. The direct deterministic discretization is not feasible for the huge computational effort. In the first successful multiscale simulation scheme—CONNFFESSIT [11]—one may simulate these elastic dumbbells in the solvent numerically according to a stochastic differential equation (SDE) such as (1.3), except that the convection term $d\boldsymbol{Q} + \boldsymbol{u} \cdot \nabla \boldsymbol{Q} dt$ is replaced by material derivative $D\boldsymbol{Q}$. It is obviously a Lagrangian method. This algorithm suffers from the local concentration and sparsity of the polymers, which makes the empirical average for polymeric stress lose accuracy. It is improved with Brownian configuration fields (BCF) [8] by introducing the fields $\boldsymbol{Q}$ in (1.3) with an Eulerian viewpoint. Some deeper explanations and further developments of BCF may be found in [3, 17].

Many mathematical issues for this type of problem have been considered in recent years. The analysis for Hookean dumbbell $\boldsymbol{F}(\boldsymbol{Q}) = \boldsymbol{Q}$ in shear flow was first considered in [9, 5]. In this simplified form the linearity and decoupling of one component of $\boldsymbol{Q}$ to the other are sufficiently utilized. The mean square convergence after excluding a set of exponentially small probability is obtained. The local well-posedness for FENE model $\boldsymbol{F}(\boldsymbol{Q}) = \boldsymbol{Q}/(1 - \frac{Q^2}{Q_0^2})$ in shear flow is obtained in [10], where Q_0 is the maximal extension of the spring. In this case, the force is singular when $\boldsymbol{Q}$ tends to Q_0. A delicate analysis for the nonexplosive behavior of $\boldsymbol{Q}$ is deduced. In a high dimensional case, the local well-posedness is proved in [6] for the coupled system under the polynomial growth condition of $\boldsymbol{Q}$

$$|\nabla^m \boldsymbol{F}(\boldsymbol{Q})| \le 1 + |\boldsymbol{Q}|^p \quad (m = 0, 1, 2, 3, 4) \tag{1.6}$$

for some positive real p. The most recent progress for the mathematical analysis of complex fluids is reviewed in [12]. To the best knowledge of the authors, there are no results concerning the convergence of the BCF methods for the dumbbell model in the nonshear flow case, even for the Hookean spring, which is the main contribution of this paper.

The basic approach we take is very similar to the shear flow case [5, 9] under the assumption that the continuous solution is smooth enough. But a stronger result is obtained for the analysis of the error of polymeric stress $\mathbb{E}(\boldsymbol{Q}\boldsymbol{Q})$. In order to exhibit the essence of the proof, we take the finite difference MAC scheme (described in section 2) for a model problem in a rectangular domain D with periodic boundary condition; this is quite common in computational fluid mechanics. The proof would be easily adapted to general boundary conditions. The first point of our proof is that the space should be discretized. In this case the inverse inequality

$$\|\bar{\boldsymbol{u}}^n\|_{L_h^\infty} \le h^{-\frac{d}{2}} \|\bar{\boldsymbol{u}}^n\|_{L_h^2}, \quad \|\bar{\boldsymbol{Q}}^n\|_{L_h^\infty} \le h^{-\frac{d}{2}} \|\bar{\boldsymbol{Q}}^n\|_{L_h^2}$$

could be applied, where $\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{Q}}^n$ are numerical solutions of the coupled system, h is spatial stepsize, d is space dimension, and L_h^∞ and L_h^2 are the discrete L^∞ and L^2 norms defined in (2.9) and (2.10). Then the discrete L^∞ norm of $\bar{\boldsymbol{u}}^n$ and $\bar{\boldsymbol{Q}}^n$ could be bounded by the discrete L^2 norm and the high order convergence of the scheme. This is essentially one part of Strang's trick [19]. Let us consider the error of $\boldsymbol{Q}$

$$\boldsymbol{E}^{n,m} := \tilde{\boldsymbol{Q}}^{n,m} - \bar{\boldsymbol{Q}}^{n,m}, \quad m = 1, 2, \ldots, N,$$

only with time discretization, where the superscript $m = 1, 2, \ldots, N$ represents N different fields driven by N independent Wiener processes in BCF methods. Here $\tilde{\boldsymbol{Q}}^{n,m}$ means the numerical solution of $\boldsymbol{Q}$ with exact velocity $\boldsymbol{u}^n = \boldsymbol{u}(t_n)$. Their rigorous definition is in (4.14) and (4.15). $\boldsymbol{E}^{n,m}$ satisfies

$$\begin{aligned} \frac{1}{\delta t}(\boldsymbol{E}^{n+1,m} - \boldsymbol{E}^{n,m}) + \bar{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{E}^{n+1,m} + \boldsymbol{e}^n \cdot \nabla \tilde{\boldsymbol{Q}}^{n+1,m} \\ = \bar{\kappa}^n \boldsymbol{E}^{n+1,m} + \nabla \boldsymbol{e}^n \tilde{\boldsymbol{Q}}^{n+1,m} - \boldsymbol{E}^{n,m}, \end{aligned} \tag{1.7}$$

where $\boldsymbol{e}^n = \boldsymbol{u}^n - \bar{\boldsymbol{u}}^n$ is the error of velocity, and $\bar{\kappa}^n = (\nabla \bar{\boldsymbol{u}}^n)^T$. The term

$$\int_D \bar{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{E}^{n+1,m} \cdot \boldsymbol{E}^{n+1,m} d\boldsymbol{x} = 0 \tag{1.8}$$

in the space continuous case. In order to estimate $\int_D \boldsymbol{e}^n \cdot \nabla \tilde{\boldsymbol{Q}}^{n+1,m} \cdot \boldsymbol{E}^{n+1,m} d\boldsymbol{x}$, one needs

$$\| \langle |\nabla \tilde{\boldsymbol{Q}}^{n+1,\cdot}|^2 \rangle_N \|_{L^\infty} \preceq Const., \tag{1.9}$$

where $\langle \cdot \rangle_N$ is the empirical average with respect to m defined in (2.7), and "$\preceq$" means "$\leq$" after excluding a set of exponentially small probability. This could be very difficult. It is easier to obtain the following L^2-type estimate:

$$\| \langle |\nabla \tilde{\boldsymbol{Q}}^{n+1,\cdot}|^2 \rangle_N \|_{L^2} \preceq Const. \tag{1.10}$$

But it cannot be transferred back to the L^∞ norm. This difficulty could be overcome by using the inverse inequality trick in the spatially discrete case.

Even if the space is discretized, the inverse inequality makes the estimate of the mean square type in [9] inapplicable. This can be clarified by the following arguments. Now the term $\bar{\boldsymbol{u}}^n \cdot \nabla \boldsymbol{E}^{n+1}$ becomes $\bar{\boldsymbol{u}}^n \cdot \nabla_h \boldsymbol{E}^{n+1}$, where ∇_h is some kind of spatially discretized derivative (for example, one of (2.1)), and we have dropped the subscripts "i,j" denoting spatial dependence of variables for simplicity. In this discrete case

$$\sum_{ij} \bar{\boldsymbol{u}}^n \cdot \nabla_h \boldsymbol{E}^{n+1,m} \cdot \boldsymbol{E}^{n+1,m} \neq 0. \tag{1.11}$$

It has only the estimate

$$\left| \sum_{ij} \bar{\boldsymbol{u}}^n \cdot \nabla_h \boldsymbol{E}^{n+1,m} \cdot \boldsymbol{E}^{n+1,m} \right| \leq \|\nabla_h \bar{\boldsymbol{u}}^n\|_{L_h^\infty} \|\boldsymbol{E}^{n+1,m}\|_{L_h^2}. \tag{1.12}$$

One demands

$$\delta t \|\nabla_h \bar{\boldsymbol{u}}^n\|_{L_h^\infty} \preceq Const. \tag{1.13}$$

This could be transferred to the L_h^2 estimate by inverse inequality, but a convergence result of the mean square type gives only

$$\mathbb{E}\Big(\delta t\|\nabla_h\bar{\boldsymbol{u}}^n\|_{L_h^\infty}\cdot\mathbf{1}_{\mathcal{A}^c}\Big)\leq Const., \tag{1.14}$$

where $\mathcal{A}$ is a set of exponentially small probability. The expectation cannot be eliminated, which means a stronger result is needed.

The key point in this paper is to prove the following large deviation type for the error of polymeric stress:

$$\|\mathbb{E}(\tilde{\boldsymbol{Q}}^{n,\cdot}\tilde{\boldsymbol{Q}}^{n,\cdot})-\langle\tilde{\boldsymbol{Q}}^{n,\cdot}\tilde{\boldsymbol{Q}}^{n,\cdot}\rangle_N\|_{L_h^\infty}^2\preceq\frac{1}{N^{1-\epsilon}}, \tag{1.15}$$

where $0<\epsilon<1$ is an arbitrary small positive real number. This goal is achieved by observing that $\boldsymbol{Q}$ is a Gaussian process and the discrete $\tilde{\boldsymbol{Q}}^{n,m}$ are independently and identically distributed (i.i.d.) Gaussian random variables in section 4.

Finally, the authors want to comment that our method may be generalized to the finite element method in principle. But in the finite element method, the treatment of the convection term must be very careful. Many more details will be involved, which is beyond the scope of the current paper.

The rest of the paper is organized as follows. In section 2, we state some notation and the main theorem. In section 3, we cite some lemmas for later use. An explicit solution and some analysis for $\boldsymbol{Q}$ are given in section 4. The final convergence analysis is given in section 5. Some technical details are included in the appendix.

2. Main results. In order to state the main theorem, we should introduce some related notation and definitions first. The boundary condition is taken to be periodic in space R^d, where $d=2,3$ is the space dimension. One period is the cube $[0,1]^d$ denoted by $D\subset R^d$. Without loss of generality, we will use only the two-dimensional notations; the three-dimensional analysis is similar. We define the continuous solution $(\boldsymbol{u},p,\boldsymbol{Q})$ and the discretized solution $(\bar{\boldsymbol{u}},\bar{p},\bar{\boldsymbol{Q}})$ as

$$\begin{aligned}\boldsymbol{u}&=(u^1,u^2),\quad \boldsymbol{Q}=(Q^1,Q^2),\\ \bar{\boldsymbol{u}}&=(\bar{u}^1,\bar{u}^2),\quad \bar{\boldsymbol{Q}}=(\bar{Q}^1,\bar{Q}^2),\end{aligned}$$

where the superscript represents the coordinate components. The difference operators to u^1 with respect to the x-direction are defined as

$$\begin{aligned}\bar{D}_xu^1(x,y)&:=\frac{u^1(x+\Delta x,y)-u^1(x-\Delta x,y)}{2\Delta x},\\ D_xu^1(x,y)&:=\frac{u^1(x+\frac{\Delta x}{2},y)-u^1(x-\frac{\Delta x}{2},y)}{\Delta x},\\ \bar{E}_xu^1(x,y)&:=\frac{u^1(x+\Delta x,y)+u^1(x-\Delta x,y)}{2},\\ E_xu^1(x,y)&:=\frac{u^1(x+\frac{\Delta x}{2},y)+u^1(x-\frac{\Delta x}{2},y)}{2},\end{aligned}$$

where Δx is the space stepsize. Similar definitions are given to the y-direction and u^2-component. Correspondingly, the discrete gradient and Laplacian operators are defined as

$$\nabla_h:=\begin{pmatrix}D_x\\ D_y\end{pmatrix},\qquad \bar{\nabla}_h:=\begin{pmatrix}\bar{D}_x\\ \bar{D}_y\end{pmatrix},\qquad \Delta_h:=D_x^2+D_y^2. \tag{2.1}$$

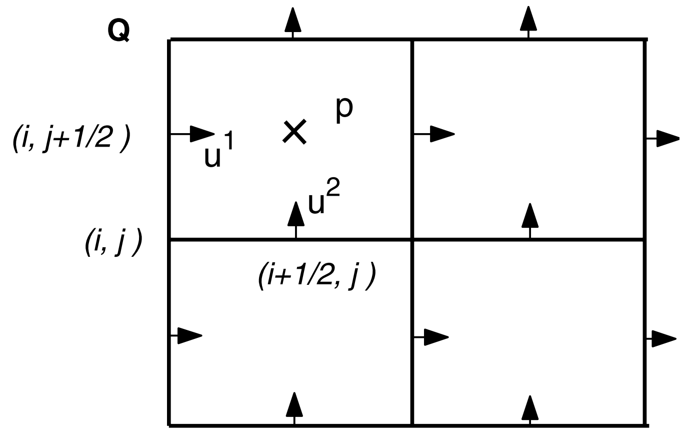

FIG. 2.1. *Staggered grid for u^1, u^2, p, and $\boldsymbol{Q}$.*

The staggered MAC grid is applied to (1.1), (1.2), and (1.3) spatially, and the discretized variables will be denoted as $\bar{u}$, $\bar{p}$, and $\bar{Q}$, respectively. As in Figure 2.1, let $x_i = i\Delta x$, $y_j = j\Delta y$ be the subdivision points in the x-axis and the y-axis, where we take $\Delta x = \Delta y = h$ to be the space stepsize for simplicity. The components of fluid velocity u^1 is defined at the points $(i, j+\frac{1}{2})$ (i.e., the nodes labelled with "$\rightarrow$"), u^2 is defined at the points $(i+\frac{1}{2}, j)$ (i.e., the nodes labelled with "$\uparrow$"), the pressure p is defined at the points $(i+\frac{1}{2}, j+\frac{1}{2})$ (i.e., the nodes labelled with "$\times$"), and the configuration vector $\boldsymbol{Q}$ is defined at the points (i, j). For the sake of simplicity, we will omit the subscript (i, j) later. All of the discretized variables will be understood at the corresponding nodes without confusion.

For the backward Euler to time and MAC scheme to space, we have the discretized hydrodynamic equations

$$\begin{aligned}(2.2)\qquad &\frac{\bar{u}^{1,n+1}-\bar{u}^{1,n}}{\delta t} + \bar{u}^{1,n}\bar{D}_x\bar{u}^{1,n} + E_xE_y\bar{u}^{2,n}\bar{D}_y\bar{u}^{1,n} + D_x\bar{p}^{n+1}\\ &\quad = \Delta_h\bar{u}^{1,n+1} + \bar{D}_xE_y\bar{\tau}^{11,n} + D_y\bar{\tau}^{21,n} \quad \text{at} \quad \left(i, j+\frac{1}{2}\right),\end{aligned}$$

$$\begin{aligned}(2.3)\qquad &\frac{\bar{u}^{2,n+1}-\bar{u}^{2,n}}{\delta t} + E_xE_y\bar{u}^{1,n}\bar{D}_x\bar{u}^{2,n} + \bar{u}^{2,n}\bar{D}_y\bar{u}^{2,n} + D_y\bar{p}^{n+1}\\ &\quad = \Delta_h\bar{u}^{2,n+1} + D_x\bar{\tau}^{12,n} + \bar{D}_yE_x\bar{\tau}^{22,n} \quad \text{at} \quad \left(i+\frac{1}{2}, j\right),\end{aligned}$$

$$(2.4)\qquad D_x\bar{u}^{1,n+1} + D_y\bar{u}^{2,n+1} = 0 \quad \text{at} \quad \left(i+\frac{1}{2}, j+\frac{1}{2}\right),$$

where $\bar{u}^{l,n}$ ($l = 1, 2$; $n = 0, 1, \ldots$) are numerical velocity at time $t_n = n\delta t$. The polymer stress $\bar{\tau}^{kl,n}$ will be defined after the discretization of $\boldsymbol{Q}$.

For the equation of $\boldsymbol{Q}$, we apply the Euler–Maruyama scheme to time, centered difference to space, and the implicit scheme for the following convection term:

$$\begin{aligned}(2.5)\qquad &\bar{Q}^{1,m,n+1} - \bar{Q}^{1,m,n} - \left(E_y\bar{u}^{1,n}\bar{D}_x\bar{Q}^{1,m,n+1} + E_x\bar{u}^{2,n}\bar{D}_y\bar{Q}^{1,m,n+1}\right)\delta t\\ &\quad = \left(\bar{D}_xE_y\bar{u}^{1,n}\bar{Q}^{1,m,n} + D_y\bar{u}^{1,n}\bar{Q}^{2,m,n}\right)\delta t - \bar{Q}^{1,m,n}\delta t + \delta W^{1,m,n},\end{aligned}$$

$$\bar{Q}^{2,m,n+1} - \bar{Q}^{2,m,n} - \left(E_y \bar{u}^{1,n} \bar{D}_x \bar{Q}^{2,m,n+1} + E_x \bar{u}^{2,n} \bar{D}_y \bar{Q}^{2,m,n+1}\right)\delta t = \left(D_x \bar{u}^{2,n} \bar{Q}^{1,m,n} + \bar{D}_y E_x \bar{u}^{2,n} \bar{Q}^{2,m,n}\right)\delta t - \bar{Q}^{2,m,n}\delta t + \delta W^{2,m,n} \quad \text{at } (i,j), \quad m = 1, \dots, N, \tag{2.6}$$

where $\delta W^{l,m,n}$ are i.i.d. $N(0, \delta t)$ random variables, which are the discretization of Wiener process. The superscript m in $\bar{Q}$ represents the mth replica induced by the i.i.d. temporal Gaussian random variable $\delta W^{l,m,n}$. That is one key ingredient in the BCF method.

Now we state the definition of polymer stress $\bar{\tau}^{kl,n}$

$$\bar{\tau}^{kl,n} = \frac{1}{N}\sum_{m=1}^{N} \bar{Q}^{k,m,n}\bar{Q}^{l,m,n} := \left\langle \bar{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\right\rangle_N, \tag{2.7}$$

which is the empirical average of $Q^k Q^l$. Here $k, l = 1, 2$ and $\langle f\rangle_N := \frac{1}{N}\sum_{m=1}^{N} f^m$ for arbitrary N-array f. The superscript "$\cdot$" means to be taken for all possible choices.

Define the error of $\boldsymbol{u}$

$$e^{l,n} = u^{l,n} - \bar{u}^{l,n}$$

on corresponding nodes $(i, j+\frac{1}{2})$ or $(i+\frac{1}{2}, j)$ for e^1 or e^2, respectively, where $u^{l,n} = u^l(t_n)$, and the notation for discrete

$$L_h^1 \text{ norm} \quad \|\bar{\boldsymbol{u}}^n\|_{L_h^1} = h^d \sum_{ij}\left(|\bar{u}^{1,n}_{i,j+\frac{1}{2}}| + |\bar{u}^{2,n}_{i+\frac{1}{2},j}|\right), \tag{2.8}$$

$$L_h^2 \text{ norm} \quad \|\bar{\boldsymbol{u}}^n\|^2_{L_h^2} = h^d \sum_{ij}\left((\bar{u}^{1,n}_{i,j+\frac{1}{2}})^2 + (\bar{u}^{2,n}_{i+\frac{1}{2},j})^2\right), \tag{2.9}$$

$$L_h^\infty \text{ norm} \quad \|\bar{\boldsymbol{u}}^n\|_{L_h^\infty} = \max_{ij}\left\{|\bar{u}^{1,n}_{i,j+\frac{1}{2}}|,\ |\bar{u}^{2,n}_{i+\frac{1}{2},j}|\right\}, \tag{2.10}$$

and

$$L_\tau^2 L_h^2 \text{ norm} \quad \|\bar{\boldsymbol{u}}^n\|^2_{L_\tau^2 L_h^2} = \delta t \sum_n \|\bar{\boldsymbol{u}}^n\|^2_{L_h^2}, \tag{2.11}$$

$$L_\tau^\infty L_h^\infty \text{ norm} \quad \|\bar{\boldsymbol{u}}^n\|_{L_\tau^\infty L_h^\infty} = \max_n \|\bar{\boldsymbol{u}}^n\|_{L_h^\infty}, \tag{2.12}$$

where d is the space dimension.

The main results of this paper are as follows.

THEOREM 2.1. *Under the condition* $\boldsymbol{u} \in C^1([0,T]\times D)$*, the SDE*

$$d\boldsymbol{Q} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{Q}dt = (\kappa\boldsymbol{Q} - \boldsymbol{Q})dt + d\boldsymbol{W}_t, \qquad \boldsymbol{Q}(\boldsymbol{x}, 0) = \boldsymbol{Q}_0(\boldsymbol{x}), \tag{2.13}$$

has the explicit solution in Lagrangian coordinates $\boldsymbol{\alpha}$ *defined in* (4.1)

$$\boldsymbol{Q}(\boldsymbol{\alpha}, t) = e^{-t}F(\boldsymbol{\alpha}, t)\boldsymbol{Q}_0(\boldsymbol{\alpha}) + F(\boldsymbol{\alpha}, t)\cdot\int_0^t e^{s-t}F^{-1}(\boldsymbol{\alpha}, s)\cdot d\boldsymbol{W}_s, \tag{2.14}$$

where $F(\boldsymbol{\alpha}, t)$ *is the deformation tensor defined in* (4.4). *With the assumption that* $\boldsymbol{Q}_0(\boldsymbol{x})$ *is a constant Gaussian random field independent of Wiener process* $\boldsymbol{W}_t$, *the following large deviation-type estimate for its numerical solution* $\tilde{\boldsymbol{Q}}^n$ *holds:*

$$\|\mathbb{E}(\tilde{\boldsymbol{Q}}^n\tilde{\boldsymbol{Q}}^n) - \langle\tilde{\boldsymbol{Q}}^n\tilde{\boldsymbol{Q}}^n\rangle_N\|^2_{L^\infty_h} \preceq \frac{1}{N^{1-\epsilon}} \tag{2.15}$$

after excluding an event $\mathcal{A}$ *with probability*

$$\frac{(d^2+d)T}{h^d\delta t}e^{-Nb} + \frac{dT}{h^d\delta t}e^{-N^\epsilon\frac{1}{2B_3}} + \frac{d^2T}{h^d\delta t}e^{-N^\epsilon B_4} \tag{2.16}$$

if $\delta t = h^2$, $N = h^{-\alpha}$ $(\alpha > 0)$, *where* b, B_3, B_4 *are defined in* (4.32), (4.36), *and* (4.42) *respectively, and* $0 < \epsilon < 1$ *is an arbitrary fixed small positive number. The rigorous definition of* $\tilde{\boldsymbol{Q}}$ *is in* (4.14) *and* (4.15).

REMARK 2.1. *The initial condition* $\boldsymbol{Q}_0(\boldsymbol{x})$ *being a constant Gaussian random field means that* $\boldsymbol{Q}_0(\boldsymbol{x}) = (Q_0^1, Q_0^2, \ldots, Q_0^d)$, *and* Q_0^l $(l = 1, 2, \ldots, d)$ *are independent Gaussian random variables. This is reasonable when polymers are initially at equilibrium* [9].

THEOREM 2.2. *Assume that*

$$\boldsymbol{u} \in C^5([0,T] \times D). \tag{2.17}$$

If $\delta t = h^2$, $N = h^{-\alpha}$, $\alpha > d$, *where* d *is the space dimension* $(d = 2, 3)$, *then we have the following error estimates of BCF methods for the Hookean dumbbell model with finite difference method:*

$$\|\boldsymbol{e}^{\cdot,n}\|^2_{L^\infty_\tau L^2_h} \preceq C\left(\delta t^2 + h^4 + \frac{1}{N^{1-\epsilon}}\right) \tag{2.18}$$

and

$$\|\nabla_h\boldsymbol{e}^{\cdot,n}\|^2_{L^2_\tau L^2_h} \preceq C\left(\delta t^2 + h^4 + \frac{1}{N^{1-\epsilon}}\right) \tag{2.19}$$

after excluding the same event $\mathcal{A}$ *as that in Theorem* 2.1, *where* C *depends on the norm* $\|\boldsymbol{u}\|_{C^5(D\times[0,T])}$.

REMARK 2.2. *The conditions* $\delta t = h^2$, $N = h^{-\alpha}$, *and* $\alpha > d$ *are much more stringent than those in the shear flow case* [9]. *But it is necessary for the current proof because of the application of inverse inequality. In the shear flow case* [9], *the choice of* δt, h, *and* N *could be independent of each other, and the probability of excluded event takes the form similar to* $\frac{T}{\delta t}e^{-\frac{N}{\delta t} - N\ln\delta t}$. *It is easy to find that the probability tends to* 0 *whenever* $\delta t \to 0$ *or* $N \to \infty$ *and the other is fixed. It is still an issue on how to remove the dependent condition on* h, δt, *and* N *in the current proof.*

REMARK 2.3. *Though we prove the convergence of the numerical scheme to the continuous solution, this does not imply another existence proof in the continuous level. The proof relies heavily on the existence of a priori smooth solution, while this could be proved for the nonlinear dumbbell model with polynomial growth condition in local time* [6]. *The convergence does not hold only in local time; it is valid until the solution loses its smoothness demanded in the proof.*

Notation. We will use the shorthand $a \preceq b$ in the rest of this paper to denote $a \leq b$, except for an event of small probability approaching zero exponentially as that in the paper [14].

3. Preliminaries and lemmas. Here we present some technical lemmas without proof for later use.

LEMMA 3.1 (inverse inequality for spatial discretization).

$$\|\bar{\boldsymbol{u}}\|_{L_h^\infty} \le h^{-\frac{d}{2}}\|\bar{\boldsymbol{u}}\|_{L_h^2}, \quad \|\nabla_h\bar{\boldsymbol{u}}\|_{L_h^2} \le h^{-1}\|\bar{\boldsymbol{u}}\|_{L_h^2}, \tag{3.1}$$

where d is the dimension, and all the norms and variables are spatially discretized.

LEMMA 3.2 (discrete integration by parts).

$$\sum_{ij} \bar{\boldsymbol{u}} \cdot \nabla_h \bar{p} = -\sum_{ij} \bar{p}\nabla_h \cdot \bar{\boldsymbol{u}}, \tag{3.2}$$

$$h^d \sum_{ij} \Delta_h \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{u}} = -\|\nabla_h \bar{\boldsymbol{u}}\|^2_{L_h^2}. \tag{3.3}$$

LEMMA 3.3 (discrete Gronwall inequality). *If $a_i \ge 0$, $c_i \ge 0$ $(i = 0, \dots, n)$, which satisfy*

$$a_n \le (1 + c_n\delta t)a_{n-1} + \delta t^p \tag{3.4}$$

and $\delta t \sum_{i=1}^n c_i \le C_0$, $n\delta t \le T$, then we have the following inequality:

$$a_n \le C_1 a_0 + C_2 \delta t^{p-1}, \tag{3.5}$$

where $C_1 = e^{C_0}$, $C_2 = Te^{C_0}$.

LEMMA 3.4 (Cramér's theorem). *Let $\{X_n\}_1^N$ be P-i.i.d. random variables with common distribution μ, assume the associated moment generating function M_μ satisfies*

$$M_\mu(\lambda) := \int_R e^{\lambda x}\mu(dx) < \infty \quad \text{for all } \lambda \in R, \tag{3.6}$$

set $m = \int_R x\mu(dx)$, and define I_μ as the Legendre transform

$$I_\mu(x) := \sup\{\lambda x - \Lambda_\mu(\lambda) : \lambda \in R\}, \qquad x \in R,$$

of $\Lambda_\mu(\lambda)$, where $\Lambda_\mu(\lambda) = \log(M_\mu(\lambda))$ is the logarithmic moment generating function of μ; then

$$P\Big(\langle X\rangle_N \ge a\Big) \le e^{-NI_\mu(a)} \quad \text{for all } a \in [m, \infty), \tag{3.7}$$

$$P\Big(\langle X\rangle_N \le a\Big) \le e^{-NI_\mu(a)} \quad \text{for all } a \in (-\infty, m]. \tag{3.8}$$

The proof of this lemma may be found in [20].

4. Analysis of $\boldsymbol{Q}$'s equation. In this section, we will give an explicit solution for the equation of $\boldsymbol{Q}$ and consider some numerical issues with the velocity term $\boldsymbol{u}$ being exact deterministic variables.

4.1. An explicit solution of linear Hookean dumbbell model. Let us first consider the linear Hookean dumbbell model (2.13) with $\boldsymbol{u}$ known. We assume the initial value $\boldsymbol{Q}_0(\boldsymbol{x})$ is as described in Remark 2.1 and the covariance matrix of $\boldsymbol{Q}_0$ is Σ_0. We assume Σ_0 is a symmetric positive definite matrix.

LEMMA 4.1. *The SDE* (2.13) *has the explicit solution* (2.14).

Proof. Define the flow map

$$\frac{d\boldsymbol{x}(\boldsymbol{\alpha},t)}{dt} = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{\alpha},t),t), \qquad \boldsymbol{x}(\boldsymbol{\alpha},0) = \boldsymbol{\alpha}, \tag{4.1}$$

and let $\boldsymbol{Q}(\boldsymbol{\alpha},t) = \boldsymbol{Q}(\boldsymbol{x}(\boldsymbol{\alpha},t),t)$; then $\boldsymbol{Q}(\boldsymbol{\alpha},t)$ satisfies

$$d\boldsymbol{Q} = (\kappa\boldsymbol{Q} - \boldsymbol{Q})dt + d\boldsymbol{W}_t, \qquad \boldsymbol{Q}(\boldsymbol{\alpha},0) = \boldsymbol{Q}_0(\boldsymbol{\alpha}). \tag{4.2}$$

The smoothness and incompressibility of $\boldsymbol{u}(\boldsymbol{x},t)$ guarantee the flow map is a homeomorphism from R^d to R^d. Define the solution operator of the flow map $\mathcal{F}_s^t : R^d \to R^d$, i.e.,

$$\frac{d\mathcal{F}_s^t\boldsymbol{x}}{dt} = \boldsymbol{u}(\mathcal{F}_s^t\boldsymbol{x},t), \qquad \mathcal{F}_s^s\boldsymbol{x} = \boldsymbol{x};$$

then

$$\boldsymbol{x}(\boldsymbol{\alpha},t) = \mathcal{F}_0^t\boldsymbol{\alpha}. \tag{4.3}$$

In order to give the explicit solution of (4.2), we define the deformation tensor

$$F(\boldsymbol{\alpha},t) = \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{\alpha}}, \quad \text{i.e.,} \quad F_{ij} = \frac{\partial x_i}{\partial\alpha_j}, \tag{4.4}$$

motivated by [13]. From the incompressibility condition of $\boldsymbol{u}$ and $F(\boldsymbol{\alpha},0) = I$, we have

$$\det F(\boldsymbol{\alpha},t) = 1, \tag{4.5}$$

and F, F^{-1} satisfies

$$\frac{dF}{dt} = \kappa\cdot F, \qquad \frac{dF^{-1}}{dt} = -F^{-1}\cdot\kappa, \tag{4.6}$$

respectively.

Define $\boldsymbol{P}(\boldsymbol{\alpha},t) = e^tF^{-1}\cdot\boldsymbol{Q}$; then

$$d\boldsymbol{P}(\boldsymbol{\alpha},t) = (e^tF^{-1}\cdot\boldsymbol{Q} - e^tF^{-1}\cdot\kappa\boldsymbol{Q})dt + e^tF^{-1}\cdot d\boldsymbol{Q} = e^tF^{-1}\cdot d\boldsymbol{W}_t, \tag{4.7}$$

so we have the explicit solution of (4.2)

$$\boldsymbol{Q}(\boldsymbol{\alpha},t) = e^{-t}F(\boldsymbol{\alpha},t)\boldsymbol{Q}_0(\boldsymbol{\alpha}) + F(\boldsymbol{\alpha},t)\cdot\int_0^t e^{s-t}F^{-1}(\boldsymbol{\alpha},s)\cdot d\boldsymbol{W}_s. \tag{4.8}$$

Pushing forward to the Eulerian coordinates we will have the corresponding solution

$$\boldsymbol{Q}(\boldsymbol{x},t) = e^{-t}F(\boldsymbol{x},t)\boldsymbol{Q}_0(\boldsymbol{x}) + F(\boldsymbol{x},t)\cdot\int_0^t e^{s-t}F^{-1}(\mathcal{F}_t^s\boldsymbol{x},s)\cdot d\boldsymbol{W}_s. \tag{4.9}$$

From this explicit form and the assumption on $\boldsymbol{Q}_0(\boldsymbol{x})$ we have that $\boldsymbol{Q}(\boldsymbol{\alpha},t)$ is a spatially smooth Gaussian random field $N(0,\Sigma(\boldsymbol{\alpha},t))$ and

$$\begin{aligned}\Sigma(\boldsymbol{\alpha},t) = \mathbb{E}\boldsymbol{Q}\boldsymbol{Q}^T &= e^{-2t}F\cdot\Sigma_0\cdot F^T \\ &+ F(\boldsymbol{\alpha},t)\cdot\int_0^t e^{2(s-t)}F^{-1}F^{-T}(\boldsymbol{\alpha},s)ds\cdot F^T(\boldsymbol{\alpha},t).\end{aligned} \tag{4.10}$$

It is clear that $\Sigma(\boldsymbol{\alpha}, t)$ is also symmetric positive definite. ∎

LEMMA 4.2. *The spectrum of covariance matrix $\Sigma(\boldsymbol{x}, t)$ has the following uniform bounds:*

$$0 < c_0 \leq \lambda_{\min}(\Sigma(\boldsymbol{x}, t)) \leq \lambda_{\max}(\Sigma(\boldsymbol{x}, t)) \leq C_0, \tag{4.11}$$

where the constants C_0 and c_0 depend only on $\|\boldsymbol{u}\|_{C^1(D\times[0,T])}$ and Σ_0. Thus the diagonal entry $\sigma_i^2(\boldsymbol{x}, t)$ of $\Sigma(\boldsymbol{x}, t)$ has the corresponding lower bound c_0 and upper bound C_0.

Proof. From the Wieland–Hoffman theorem [7], the ith eigenvalue of any symmetric matrix is continuous with respect to the symmetric perturbations. Then from expression (4.10) all of the eigenvalues of $\Sigma(\boldsymbol{\alpha}, t)$ are a continuous function of $\boldsymbol{\alpha}$ and t. Because $(\boldsymbol{\alpha}, t) \in D \times [0, T]$ is a compact region, $\lambda_{\min}$ achieves a lower bound c_0 and $\lambda_{\max}$ achieves an upper bound C_0. From the positivity of $\Sigma(\boldsymbol{\alpha}, t)$, c_0 is positive.

From the Courant–Fisher theorem, the diagonal entry $\sigma_i^2(\boldsymbol{\alpha}, t)$ of $\Sigma(\boldsymbol{\alpha}, t)$ has the same lower bound and upper bound. Pushing forward to the Eulerian coordinates gives the desired results. ∎

LEMMA 4.3. *If $\boldsymbol{u} \in C^{m+1}(D \times [0, T])$ and $\boldsymbol{Q}_0 \in C^m(D \times [0, T], L^2(\Omega))$, then*

$$\nabla^m \boldsymbol{Q} \in L^\infty(D \times [0, T], L^2(\Omega)) \tag{4.12}$$

for arbitrary $m \in \mathbb{N} \cup \{0\}$, and

$$\nabla^m \partial_t \boldsymbol{Q} \in L^\infty(D \times [0, T], L^2(\Omega)) \tag{4.13}$$

for arbitrary $m \in \mathbb{N}$.

Proof. From the explicit form of $\boldsymbol{Q}(\boldsymbol{x}, t)$ and the smoothness of $\boldsymbol{u}$ and $\boldsymbol{Q}_0$, the inequality (4.12) above is obvious by direct differentiation and Itô isometry [15]. Inequality (4.13) comes from the SDE (2.13) and the fact that the spatial derivative of $\boldsymbol{W}_t$ is 0; thus the most singular term vanishes.

Another estimating method is to differentiate both sides of (2.13) and perform an energy estimate as in [6]. ∎

4.2. Time discretization of $\boldsymbol{Q}$. For the analysis of the error of $\boldsymbol{Q}$ we define an auxiliary random variable $\tilde{\boldsymbol{Q}}$ which obeys similar equations as (2.5) and (2.6), except the velocity $\bar{\boldsymbol{u}}$ is replaced by the exact $\boldsymbol{u}$

$$\begin{aligned} \tilde{Q}^{1,n+1} - \tilde{Q}^{1,n} - \Big(E_y u^{1,n} \bar{D}_x \tilde{Q}^{1,n+1} + E_x u^{2,n} \bar{D}_y \tilde{Q}^{1,n+1}\Big)\delta t \\ = \Big(\bar{D}_x E_y u^{1,n} \tilde{Q}^{1,n} + D_y u^{1,n} \tilde{Q}^{2,n}\Big)\delta t - \tilde{Q}^{1,n}\delta t + \delta W^{1,n}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \tilde{Q}^{2,n+1} - \tilde{Q}^{2,n} - \Big(E_y u^{1,n} \bar{D}_x \tilde{Q}^{2,n+1} + E_x u^{2,n} \bar{D}_y \tilde{Q}^{2,n+1}\Big)\delta t \\ = \Big(D_x u^{2,n} \tilde{Q}^{1,n} + \bar{D}_y E_x u^{2,n} \tilde{Q}^{2,n}\Big)\delta t - \tilde{Q}^{2,n}\delta t + \delta W^{2,n}. \end{aligned} \tag{4.15}$$

This is nothing but the time discretization of $\boldsymbol{Q}$.

Defining the error

$$\tilde{E}^{l,n} = Q^{l,n} - \tilde{Q}^{l,n},$$

we have the following lemma.

LEMMA 4.4 (mean square convergence of $\tilde{\boldsymbol{Q}}$ to $\boldsymbol{Q}$).

$$\mathbb{E}\|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2 \leq C(\delta t^2 + h^4),$$

where C depends on $\|\boldsymbol{u}\|_{C^4(D\times[0,T])}$.

Proof. In order to prove the convergence theorem above, we integrate both sides of $\boldsymbol{Q}$'s equation from t_n to t_{n+1} and in the rectangle $[x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$; then we have

$$\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}} \boldsymbol{Q}(x,y,t_{n+1})dydx - \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}} \boldsymbol{Q}(x,y,t_n)dydx$$
$$= \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}} (-\boldsymbol{u}\cdot\nabla\boldsymbol{Q} + \kappa\boldsymbol{Q} - \boldsymbol{Q})dtdydx + 4h^2\cdot\delta\boldsymbol{W}^n.$$

Here $\delta\boldsymbol{W}^n$ are i.i.d. $N(0,\delta t)$ random variables in R^d.

We rewrite the above equations with the similar form as (2.5) and (2.6); then we will have the following equations and the remainder terms:

$$Q^{1,n+1} - Q^{1,n} - \Big(E_y u^{1,n}\bar{D}_x Q^{1,n+1} + E_x u^{2,n}\bar{D}_y Q^{1,n+1}\Big)\delta t \tag{4.16}$$
$$= \Big(\bar{D}_x E_y u^{1,n} Q^{1,n} + D_y u^{1,n} Q^{2,n}\Big)\delta t - Q^{1,n}\delta t + \delta W^{1,n} + R^{1,n},$$
$$Q^{2,n+1} - Q^{2,n} - \Big(E_y u^{1,n}\bar{D}_x Q^{2,n+1} + E_x u^{2,n}\bar{D}_y Q^{2,n+1}\Big)\delta t \tag{4.17}$$
$$= \Big(D_x u^{2,n} Q^{1,n} + \bar{D}_y E_x u^{2,n} Q^{2,n}\Big)\delta t - Q^{2,n}\delta t + \delta W^{2,n} + R^{2,n} \quad \text{at } (i,j),$$

where $Q^{l,n+1} = Q^l(t_{n+1})$, $l = 1,2$. The truncation errors $R^{l,n}$ are analyzed in the appendix in detail. Defining $\boldsymbol{R}^n = (R^{1,n}, R^{2,n})$, we have from the appendix

$$\boldsymbol{R}^n = \boldsymbol{R}_1^n + \boldsymbol{R}_2^n. \tag{4.18}$$

$\boldsymbol{R}_i^n$ satisfy the following estimates:

$$\mathbb{E}\|\boldsymbol{R}_1^n\|_{L_h^2}^2 \le C_1\delta t^3, \qquad \mathbb{E}\|\boldsymbol{R}_2^n\|_{L_h^2}^2 \le C_2\delta t^2(h^4 + \delta t^2), \tag{4.19}$$

where C_1 and C_2 are positive constants depending on $\|\boldsymbol{u}\|_{C^4(D\times[0,T])}$, and

$$\mathbb{E}\boldsymbol{R}_1^n\cdot\tilde{\boldsymbol{E}}^n = 0 \tag{4.20}$$

by independence.

Subtracting (4.16), (4.17) and (4.14), (4.15), we have

$$\tilde{E}^{1,n+1} - \tilde{E}^{1,n} - \Big(E_y u^{1,n}\bar{D}_x \tilde{E}^{1,n+1} + E_x u^{2,n}\bar{D}_y \tilde{E}^{1,n+1}\Big)\delta t \tag{4.21}$$
$$= \Big(\bar{D}_x E_y u^{1,n}\tilde{E}^{1,n} + D_y u^{1,n}\tilde{E}^{2,n}\Big)\delta t - \tilde{E}^{1,n}\delta t + R^{1,n},$$
$$\tilde{E}^{2,n+1} - \tilde{E}^{2,n} - \Big(E_y u^{1,n}\bar{D}_x \tilde{E}^{2,n+1} + E_x u^{2,n}\bar{D}_y \tilde{E}^{2,n+1}\Big)\delta t \tag{4.22}$$
$$= \Big(D_x u^{2,n}\tilde{E}^{1,n} + \bar{D}_y E_x u^{2,n}\tilde{E}^{2,n}\Big)\delta t - \tilde{E}^{2,n}\delta t + R^{2,n}.$$

Timing both sides with $\tilde{E}^{l,n+1}$ and taking summation on the nodes, the term

$$\sum_{ij,l}(\tilde{E}^{l,n+1} - \tilde{E}^{l,n})\cdot\tilde{E}^{l,n+1} = \frac{1}{2}\Big(\|\tilde{\boldsymbol{E}}^{n+1}\|_{L_h^2}^2 - \|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2 + \|\tilde{\boldsymbol{E}}^{n+1} - \tilde{\boldsymbol{E}}^n\|_{L_h^2}^2\Big).$$

The second term can be controlled as follows through summation by parts:

$$\sum_{ij,l}\Big(E_y u^{1,n}\bar{D}_x\tilde{E}^{l,n+1}+E_x u^{2,n}\bar{D}_y\tilde{E}^{l,n+1}\Big)\cdot\tilde{E}^{l,n+1}\le\|\nabla_h\boldsymbol{u}^n\|_{L_h^\infty}\|\tilde{\boldsymbol{E}}^{n+1}\|_{L_h^2}^2.$$

The third term can be controlled as follows through Cauchy's inequality:

$$\sum_{ij}\Big(\bar{D}_xE_yu^{1,n}\tilde{E}^{1,n}+D_yu^{1,n}\tilde{E}^{2,n}\Big)\cdot\tilde{E}^{1,n+1}+\Big(D_xu^{2,n}\tilde{E}^{1,n}$$
$$+\bar{D}_yE_xu^{2,n}\tilde{E}^{2,n}\Big)\cdot\tilde{E}^{2,n+1}\le\|\nabla_h\boldsymbol{u}^n\|_{L_h^\infty}^2\|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2+\|\tilde{\boldsymbol{E}}^{n+1}\|_{L_h^2}^2.$$

The fourth term is standard:

$$\sum_{ij,l}\tilde{E}^{l,n}\cdot\tilde{E}^{l,n+1}\le\frac{1}{2}\Big(\|\tilde{\boldsymbol{E}}^{n+1}\|_{L_h^2}^2+\|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2\Big).$$

The term $\boldsymbol{R}^n\cdot\tilde{\boldsymbol{E}}^{n+1}$ can be estimated by

$$\boldsymbol{R}^n\cdot\tilde{\boldsymbol{E}}^{n+1}\le\boldsymbol{R}^n\cdot\tilde{\boldsymbol{E}}^n+\frac{1}{2}(|\tilde{\boldsymbol{E}}^{n+1}-\tilde{\boldsymbol{E}}^n|^2)+\frac{1}{2}|\boldsymbol{R}^n|^2$$

and $\boldsymbol{R}^n\cdot\tilde{\boldsymbol{E}}^n=\boldsymbol{R}_1^n\cdot\tilde{\boldsymbol{E}}^n+\boldsymbol{R}_2^n\cdot\tilde{\boldsymbol{E}}^n$,

$$\boldsymbol{R}_2^n\cdot\tilde{\boldsymbol{E}}^n\le\frac{1}{2\delta t}|\boldsymbol{R}_2^n|^2+\delta t|\tilde{\boldsymbol{E}}^n|^2.$$

Combining all terms together and taking expectation, by using the important identity (4.20) we have

$$\mathbb{E}\|\tilde{\boldsymbol{E}}^{n+1}\|_{L_h^2}^2\le(1+C_3\delta t)\ \mathbb{E}\|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2+C_4\delta t(\delta t^2+h^4),$$

where $C_3>0$ depends only on $\|\boldsymbol{u}\|_{C^1(D\times[0,T])}$, and $C_4>0$ depends on $\|\boldsymbol{u}\|_{C^4(D\times[0,T])}$.

By the discrete Gronwall inequality we obtain the desired result. ∎

LEMMA 4.5 (mean square convergence of $\bar{\nabla}_h\tilde{\boldsymbol{Q}}$ to $\bar{\nabla}_h\boldsymbol{Q}$).

$$\mathbb{E}\|\bar{\nabla}_h\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2\le C(\delta t^2+h^4),$$

where C depends on $\|\boldsymbol{u}\|_{C^5(D\times[0,T])}$.

Proof. The procedure is almost the same as Lemma 4.4. Note that for any two arrays f and g, we have

$$\bar{D}_x(fg)_{ij}=(\bar{D}_xf)_{ij}\cdot g_{i+1,j}+f_{i-1,j}\cdot(\bar{D}_xg)_{ij},$$
$$\bar{D}_y(fg)_{ij}=(\bar{D}_yf)_{ij}\cdot g_{i,j+1}+f_{i,j-1}\cdot(\bar{D}_yg)_{ij}.$$

We take $\bar{D}_x,\bar{D}_y$ to both sides of (4.21) and (4.22) and define $\tilde{F}^{l,n}=\bar{D}_x\tilde{E}^{l,n}$, $\tilde{G}^{l,n}=\bar{D}_y\tilde{E}^{l,n}$; then

$$\tilde{F}^{1,n+1}-\tilde{F}^{1,n}-\Big(E_yu_{i-1,j}^{1,n}\bar{D}_x\tilde{F}^{1,n+1}+E_xu_{i-1,j}^{2,n}\bar{D}_y\tilde{F}^{1,n+1}\Big)\delta t \tag{4.23}$$
$$+\Big(\bar{D}_xE_yu^{1,n}\tilde{F}_{i+1,j}^{1,n+1}+\bar{D}_xE_xu^{2,n}\tilde{G}_{i+1,j}^{1,n+1}\Big)\delta t=\Big(\bar{D}_xE_yu_{i-1,j}^{1,n}\tilde{F}^{1,n}+D_yu_{i-1,j}^{1,n}\tilde{F}^{2,n}\Big)\delta t$$
$$+\Big(\bar{D}_x\bar{D}_xE_yu^{1,n}\tilde{E}_{i+1,j}^{1,n}+\bar{D}_xD_yu^{1,n}\tilde{E}_{i+1,j}^{2,n}\Big)\delta t-\tilde{F}^{1,n}\delta t+\bar{D}_xR^{l,n},$$

$$\tilde{F}^{2,n+1}-\tilde{F}^{2,n}-\Big(E_yu_{i-1,j}^{1,n}\bar{D}_x\tilde{F}^{2,n+1}+E_xu_{i-1,j}^{2,n}\bar{D}_y\tilde{F}^{2,n+1}\Big)\delta t \tag{4.24}$$
$$+\Big(\bar{D}_xE_yu^{1,n}\tilde{F}_{i+1,j}^{2,n+1}+\bar{D}_xE_xu^{2,n}\tilde{G}_{i+1,j}^{2,n+1}\Big)\delta t=\Big(D_xu_{i-1,j}^{2,n}\tilde{F}^{1,n}+\bar{D}_yE_xu_{i-1,j}^{2,n}\tilde{F}^{2,n}\Big)\delta t$$
$$+\Big(\bar{D}_xD_xu^{2,n}\tilde{E}_{i+1,j}^{1,n}+\bar{D}_x\bar{D}_yE_xu^{2,n}\tilde{E}_{i+1,j}^{2,n}\Big)\delta t-\tilde{F}^{2,n}\delta t+\bar{D}_xR^{2,n},$$

and $\tilde{G}^{l,n}$ satisfies similar equations, which we will omit here. The new remainder terms $\bar{D}_x R^{l,n}, \bar{D}_y R^{l,n}$ are analyzed in the appendix.

Timing both sides with $\tilde{F}^{l,n+1}$ (or $\tilde{G}^{l,n+1}$ for $\tilde{G}^{l,n+1}$'s equation) and taking summation on the nodes, we have the following estimate with similar arguments as Lemma 4.4:

$$\mathbb{E}\|\bar{\nabla}_h \tilde{\boldsymbol{E}}^{n+1}\|_{L_h^2}^2 \le (1 + C_1\delta t)\ \mathbb{E}\|\bar{\nabla}_h \tilde{\boldsymbol{E}}^{n}\|_{L_h^2}^2 + C_2\delta t(\delta t^2 + h^4),$$

where $C_1 > 0$ depends only on $\|\boldsymbol{u}\|_{C^1(D\times[0,T])}$, and $C_2 > 0$ depends on $\|\boldsymbol{u}\|_{C^5(D\times[0,T])}$.

The discrete Gronwall inequality gives the desired result. ◻

4.3. Large deviations related to $\tilde{\boldsymbol{Q}}$. For the analysis of the error caused by $\bar{\boldsymbol{Q}}$ in BCF methods, we introduce N i.i.d. duplications of (4.14) and (4.15)

$$\tilde{Q}^{1,m,n+1} - \tilde{Q}^{1,m,n} - \Big(E_y u^{1,n} \bar{D}_x \tilde{Q}^{1,m,n+1} + E_x u^{2,n} \bar{D}_y \tilde{Q}^{1,m,n+1}\Big)\delta t = \Big(\bar{D}_x E_y u^{1,n} \tilde{Q}^{1,m,n} + D_y u^{1,n} \tilde{Q}^{2,m,n}\Big)\delta t - \tilde{Q}^{1,m,n}\delta t + \delta W^{1,m,n}, \tag{4.25}$$

$$\tilde{Q}^{2,m,n+1} - \tilde{Q}^{2,m,n} - \Big(E_y u^{1,n} \bar{D}_x \tilde{Q}^{2,m,n+1} + E_x u^{2,n} \bar{D}_y \tilde{Q}^{2,m,n+1}\Big)\delta t = \Big(D_x u^{2,n} \tilde{Q}^{1,m,n} + \bar{D}_y E_x u^{2,n} \tilde{Q}^{2,m,n}\Big)\delta t - \tilde{Q}^{2,m,n}\delta t + \delta W^{2,m,n}, \tag{4.26}$$

where $m = 1, 2, \ldots, N$. $\delta W^{l,m,n}$ are i.i.d. $N(0, \delta t)$ random variables.

LEMMA 4.6. *If the space time stepsize satisfies*

$$\|\boldsymbol{u}\|_{C^0(D\times[0,t])} \frac{\delta t}{h} < \frac{1}{d}, \tag{4.27}$$

then (4.14) *and* (4.15) *are solvable, and the random variables $\tilde{Q}^{l,n}$ are Gaussian random variables with mean* 0.

Proof. We can define a large vector $\tilde{\boldsymbol{Q}}_n$ whose components are $\tilde{\boldsymbol{Q}}^{l,n}$ at node points and rewrite (4.14) and (4.15) in matrix form. Simply denote it as

$$A_n \cdot \tilde{\boldsymbol{Q}}_{n+1} = B_n \cdot \tilde{\boldsymbol{Q}}_n + \delta \boldsymbol{W}_n,$$

where A_n, B_n are deterministic matrices, and $\delta\boldsymbol{W}_n$ are random vectors formed by $\delta W^{l,n}$. The diagonal of matrix A_n is 1, and each row of A_n has $2d$ off-diagonal elements, each of which can be bounded by $\|\boldsymbol{u}\|_{C^0(D\times[0,t])}\frac{\delta t}{2h}$. Thus A_n is strictly diagonally dominant; i.e., A_n is invertible. We have

$$\tilde{\boldsymbol{Q}}_{n+1} = (A_n)^{-1} B_n \cdot \tilde{\boldsymbol{Q}}_n + (A_n)^{-1}\delta\boldsymbol{W}_n.$$

By induction, we have that $\tilde{\boldsymbol{Q}}_n$ are Gaussian random variables because $\tilde{\boldsymbol{Q}}_0$ and $\delta\boldsymbol{W}_n$ are i.i.d. Gaussian. Taking expectation shows $\mathbb{E}\tilde{\boldsymbol{Q}}_n = 0$ immediately. ◻

LEMMA 4.7. *Suppose the random variable $\tilde{Q} \sim N(0, \tilde{\sigma}^2)$; then $\tilde{Q}^2$ obeys large deviation theory with rate function*

$$I(x) = \frac{x}{2\tilde{\sigma}^2} - \frac{1}{2} + \frac{1}{2}\ln\frac{\tilde{\sigma}^2}{x} \tag{4.28}$$

for $x > 0$.

LEMMA 4.8. *Suppose the random vector $(\tilde{Q}^1, \tilde{Q}^2) \sim N(0, \tilde{\Sigma})$, where*

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{21} & \tilde{\sigma}_{22} \end{pmatrix}.$$

$\tilde{\Sigma}$ is symmetric positive definite. Then $\tilde{Q}_1\tilde{Q}_2$ obeys large deviation theory with rate function

$$I(x) = \lambda(x)x - \Lambda(\lambda(x)),$$

for $x \in R$, where $\lambda(x)$ satisfies

$$\frac{\tilde{\sigma}_{12} + \lambda(x)\det(\tilde{\Sigma})}{1 - 2\lambda(x)\tilde{\sigma}_{12} - \lambda^2(x)\det(\tilde{\Sigma})} = x \tag{4.29}$$

and

$$\Lambda(\lambda) = -\frac{1}{2}\ln\Big(1 - 2\lambda\tilde{\sigma}_{12} - \lambda^2\det(\tilde{\Sigma})\Big). \tag{4.30}$$

The proofs of the two lemmas above are deferred to the appendix.

For later use, we have the following estimates of large deviation type after excluding a set of exponentially small probability.

LEMMA 4.9. *We have the following bounds hold for all $l = 1, 2$ and $0 \le n \le \frac{T}{\delta t}$ if $\delta t = h^2$, $N = h^{-\alpha}$, and h is small enough such that (4.27) is satisfied and the following probability of excluded set is less than 1:*

$$\|\langle|\tilde{Q}^{l,\cdot,n}|^2\rangle_N\|_{L_h^\infty} \preceq B_1, \quad \|\langle|\bar{\nabla}_h\tilde{Q}^{l,\cdot,n}|^2\rangle_N\|_{L_h^\infty} \preceq B_2, \tag{4.31}$$

after excluding a set with probability $\frac{(d^2+d)T}{h^d\delta t}e^{-Nb}$, where b is defined in (4.32), $B_1 = 4(\|\mathbb{E}\boldsymbol{Q}^2\|_{L^\infty(D\times[0,T])} + 1)$, and $B_2 = 4(\|\mathbb{E}|\nabla\boldsymbol{Q}|^2\|_{L^\infty(D\times[0,T])} + 1)$.

Proof. According to Lemma 4.6, the random variables $\tilde{Q}^{l,m,n}$ and $\bar{\nabla}_h\tilde{Q}^{l,m,n}$ are i.i.d. Gaussian random variables with mean 0. From Cramér's theorem (Lemma 3.4)

$$P\Big(\langle|\tilde{Q}^{l,\cdot,n}|^2\rangle_N \ge 2\tilde{\sigma}_l^2\Big) \le \exp(-N\cdot I(2\tilde{\sigma}_l^2)),$$

where $\tilde{\sigma}_l^2$ is the variance of $\tilde{Q}^{l,n}$, and the rate function is shown in Lemma 4.7. We have

$$I(2\tilde{\sigma}_l^2) = \frac{1}{2}(1 - \ln 2) := b > 0. \tag{4.32}$$

Because the estimate above is only for one node point in D, we should prove that $\tilde{\sigma}_1^2$ has a uniform upper bound for all node points. By triangle inequality and inverse inequality

$$\begin{aligned}\tilde{\sigma}_l^2 &= \mathbb{E}(\tilde{Q}^{l,n})^2 \le 2\Big(\mathbb{E}(Q^{l,n})^2 + \mathbb{E}(\tilde{E}^{l,n})^2\Big)\\ &\le 2\Big(\|\mathbb{E}\boldsymbol{Q}^2\|_{L^\infty(D\times[0,T])} + Ch^{-d}(\delta t^2 + h^4)\Big)\\ &\le 2(\|\mathbb{E}\boldsymbol{Q}^2\|_{L^\infty(D\times[0,T])} + 1).\end{aligned}$$

So the estimate

$$\|\langle|\tilde{Q}^{l,\cdot,n}|^2\rangle_N\|_{L_h^\infty} \preceq B_1$$

is obtained after excluding a set of probability

$$\frac{dT}{h^d\delta t}\exp(-Nb).$$

Similarly as above

$$P\Big(\langle|\bar{\nabla}_h\tilde{Q}^{l,\cdot,n}|^2\rangle_N \geq 2\bar{\sigma}_l^2\Big) \leq \exp(-N\cdot I(2\bar{\sigma}_l^2)),$$

where $\bar{\sigma}_l^2$ is the variance of $\bar{\nabla}_h\tilde{Q}^{l,n}$. The uniform upper bound of $\bar{\sigma}_l^2$ is estimated through inverse inequality and Lemma 4.5:

$$\begin{aligned}\bar{\sigma}_l^2 &= \mathbb{E}(\bar{\nabla}_h\tilde{Q}^{l,n})^2 \leq 2\Big(\mathbb{E}(\bar{\nabla}_h Q^{l,n})^2 + \mathbb{E}(\bar{\nabla}_h\tilde{E}^{l,n})^2\Big)\\ &\leq 2\Big(\|\mathbb{E}|\nabla\boldsymbol{Q}|^2\|_{L^\infty(D\times[0,T])} + Ch^{-d}(\delta t^2+h^4)\Big)\\ &\leq 2(\|\mathbb{E}|\nabla\boldsymbol{Q}|^2\|_{L^\infty(D\times[0,T])}+1).\end{aligned}$$

Thus we have

$$\|\langle|\bar{\nabla}_h\tilde{Q}^{l,\cdot,n}|^2\rangle_N\|_{L_h^\infty} \preceq B_2$$

after excluding a set of probability $\frac{d^2T}{h^d\delta t}\exp(-Nb)$. ∎

LEMMA 4.10. *We have the following estimates hold for $k,l=1,2$ and $0\leq n\leq\frac{T}{\delta t}$ if $\delta t=h^2$, $N=h^{-\alpha}$, and h is small enough such that (4.27) is satisfied and the following probability of excluded set is less than 1:*

$$\|\mathbb{E}(\tilde{Q}^{k,n}\tilde{Q}^{l,n})-\langle\tilde{Q}^{k,\cdot,n}\tilde{Q}^{l,\cdot,n}\rangle_N\|_{L_h^\infty}^2 \preceq \frac{1}{N^{1-\epsilon}}, \quad k,l=1,2, \tag{4.33}$$

after excluding a set with probability $\frac{dT}{h^d\delta t}e^{-N^\epsilon\frac{1}{2B_3}}+\frac{d^2T}{h^d\delta t}e^{-N^\epsilon B_4}$, where B_3,B_4 are defined in (4.36) and (4.42), respectively. $0<\epsilon<1$ is an arbitrary fixed small positive number.

Proof. First consider $k=l$ case. Suppose $\mathbb{E}(\tilde{Q}^l)^2=\tilde{\sigma}_l^2$. From Cramér's theorem

$$\begin{aligned}P\Big(\langle(\tilde{Q}^{l,\cdot,n})^2\rangle_N \geq \tilde{\sigma}_l^2+\delta\Big) &\leq \exp(-N\cdot I(\tilde{\sigma}_l^2+\delta)),\\ P\Big(\langle(\tilde{Q}^{l,\cdot,n})^2\rangle_N \leq \tilde{\sigma}_l^2-\delta\Big) &\leq \exp(-N\cdot I(\tilde{\sigma}_l^2-\delta))\end{aligned}$$

for $0<\delta<\tilde{\sigma}_l^2$, where $I(x)$ is the rate function for $(\tilde{Q}^l)^2$. Taylor's expansion gives

$$I(\tilde{\sigma}_l^2\pm\delta)=I(\tilde{\sigma}_l^2)\pm I'(\tilde{\sigma}_l^2)\delta+\frac{1}{2}I''(\xi)\delta^2, \tag{4.34}$$

where $\xi=\tilde{\sigma}_l^2+\theta\delta$, $\theta\in[-1,1]$. From Lemma 4.7 we have

$$I(\tilde{\sigma}_l^2)=0,\quad I'(\tilde{\sigma}_l^2)=0,\quad I''(x)=\frac{1}{2x^2}. \tag{4.35}$$

Taking $\delta=N^{-\frac{1-\epsilon}{2}}$ and noting that

$$\xi^2\leq 2\tilde{\sigma}_l^4+2N^{-(1-\epsilon)}\leq\frac{B_1^2}{2}+1:=B_3, \tag{4.36}$$

we have

$$\|\mathbb{E}(\tilde{Q}^{k,n}\tilde{Q}^{l,n})-\langle\tilde{Q}^{k,\cdot,n}\tilde{Q}^{l,\cdot,n}\rangle_N\|_{L_h^\infty}^2 \preceq \frac{1}{N^{1-\epsilon}}$$

for $k = l$ after excluding a set of probability

$$\frac{dT}{h^d \delta t} \exp\left(-N^{\epsilon} \frac{1}{2B_3}\right).$$

For the $k \neq l$ case, the idea is the same. We will take $k = 1$, $l = 2$ as an example; then we have $\mathbb{E}(\tilde{Q}^1 \tilde{Q}^2) = \tilde{\sigma}_{12}$ and $\tilde{\sigma}_{kk} = \tilde{\sigma}_k^2$ $(k = 1, 2)$. The only thing left is to prove the uniform positive lower bound of $I''(\xi)$, where $\xi = \tilde{\sigma}_{12} + \theta N^{-\frac{1-\epsilon}{2}}$, $\theta \in [-1, 1]$.

It is not difficult to find that $I(x)$ is smooth for $\tilde{\sigma}_{ij}$ and x from its concrete expression. We have Taylor's expansion

$$I''(\tilde{\sigma}_{12} + y) = I''(\tilde{\sigma}_{12}) + I'''(\tilde{\xi}) y, \tag{4.37}$$

where $\tilde{\xi} = \tilde{\sigma}_{12} + \tilde{\theta} y$, $\tilde{\theta} \in [0, 1]$, $y \in [-N^{-\frac{1-\epsilon}{2}}, N^{-\frac{1-\epsilon}{2}}]$. Thus

$$I''(\tilde{\sigma}_{12} + y) \geq I''(\tilde{\sigma}_{12}) - \max_{\tilde{\xi},\ \tilde{\sigma}_{ij}} |I'''(\tilde{\xi})| N^{-\frac{1-\epsilon}{2}}. \tag{4.38}$$

It is not difficult to obtain

$$I''(\tilde{\sigma}_{12}) = \frac{1}{\det \tilde{\Sigma} + 2\tilde{\sigma}_{12}^2} = \frac{1}{(1+\rho^2)\tilde{\sigma}_1^2 \tilde{\sigma}_2^2}, \tag{4.39}$$

where ρ is the correlation coefficient for $\tilde{Q}^{1,n}$ and $\tilde{Q}^{2,n}$ and $\rho \in [-1, 1]$. The uniform positive upper bound and lower bound for $\tilde{\sigma}_1^2$ and $\tilde{\sigma}_2^2$ are easily proved from Lemmas 4.2 and 4.9. Thus $I''(\tilde{\sigma}_{12})$ has a uniform positive lower bound b_0. The function $I'''(\tilde{\sigma}_{12} + \tilde{\theta} y)$ may be viewed as a function of variables

$$(\rho, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\theta}, y) \in [-1, 1] \times [b_{11}, b_{12}] \times [b_{21}, b_{22}] \times [0, 1] \times [-N^{-\frac{1-\epsilon}{2}}, N^{-\frac{1-\epsilon}{2}}], \tag{4.40}$$

where $b_{11}, b_{12}, b_{21}, b_{22}$ are fixed positive real numbers independent of N, h, and δt. From the uniform continuity of $I'''(\cdot)$ for these variables,

$$\max_{\tilde{\xi},\ \tilde{\sigma}_{ij}} |I'''(\tilde{\xi})| \leq b_3. \tag{4.41}$$

Thus when N is sufficiently large,

$$I''(\xi) \geq b_0 - b_3 N^{-\frac{1-\epsilon}{2}} \geq \frac{b_0}{2} := B_4 > 0; \tag{4.42}$$

thus we obtain

$$\|\mathbb{E}(\tilde{Q}^{k,n} \tilde{Q}^{l,n}) - \langle \tilde{Q}^{k,\cdot,n} \tilde{Q}^{l,\cdot,n} \rangle_N \|_{L_h^\infty}^2 \preceq \frac{1}{N^{1-\epsilon}}$$

for $k \neq l$ after excluding a set of probability $\frac{d^2 T}{h^d \delta t} \exp(-N^{\epsilon} B_4)$. □

REMARK 4.1. *In the rest of the paper, all of the excluded events are the union of the excluded sets indicated in Lemmas* 4.9 *and* 4.10. *We will denote it as* $\mathcal{A}$. *All of the inequalities with notation "*$\preceq$*" in the following sections mean that they hold after excluding event* $\mathcal{A}$.

5. Convergence analysis. After the preparations in the last section, the convergence analysis is relatively standard and easy with the inverse inequality trick.

5.1. Truncation error for $\boldsymbol{u}$. For the exact solution $\boldsymbol{u}$, we have the truncated equations under the assumption $\boldsymbol{u} \in C^2([0,T], C^4(D))$

$$\frac{u^{1,n+1}-u^{1,n}}{\delta t} + u^{1,n}\bar{D}_x u^{1,n} + E_xE_y u^{2,n}\bar{D}_y u^{1,n} + D_x p^{n+1} \tag{5.1}$$

$$= \Delta_h u^{1,n+1} + \bar{D}_x E_y \tau^{11,n} + D_y \tau^{21,n} + O(\delta t + h^2) \quad \text{at} \quad \left(i, j+\frac{1}{2}\right),$$

$$\frac{u^{2,n+1}-u^{2,n}}{\delta t} + E_xE_y u^{1,n}\bar{D}_x u^{2,n} + u^{2,n}\bar{D}_y u^{2,n} + D_y p^{n+1} \tag{5.2}$$

$$= \Delta_h u^{2,n+1} + D_x \tau^{12,n} + \bar{D}_y E_x \tau^{22,n} + O(\delta t + h^2) \quad \text{at} \quad \left(i+\frac{1}{2}, j\right),$$

$$D_x u^{1,n+1} + D_y u^{2,n+1} = O(h^2) \quad \text{at} \quad \left(i+\frac{1}{2}, j+\frac{1}{2}\right), \tag{5.3}$$

where

$$u^{l,n} = u^l(t_n), \quad \tau^{kl,n} = \mathbb{E}\big(Q^k(t_n)Q^l(t_n)\big), \qquad t_n = n\delta t,$$

and they take values at corresponding nodes as in (2.2), (2.3), and (2.4).

Because $D_x u^{1,n+1} + D_y u^{2,n+1} = O(h^2)$ in (5.3), which will bring trouble to the analysis, we need the following lemma [18].

LEMMA 5.1. *There exists $\boldsymbol{u}_h(\boldsymbol{x},t)$, such that*

$$\boldsymbol{u}_h(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) + O(h^2) \tag{5.4}$$

and $\nabla_h \cdot \boldsymbol{u}_h = 0$, where $\nabla_h \cdot \boldsymbol{u}(\boldsymbol{x},t) = D_x u^1(\boldsymbol{x},t) + D_y u^2(\boldsymbol{x},t)$.

Proof. Note that in two dimensions $\boldsymbol{u} = (-\partial_y \psi, \partial_x \psi)$, where ψ is the stream function, so we define

$$\boldsymbol{u}_h = (-D_y\psi, D_x\psi).$$

It is obvious that

$$\boldsymbol{u}_h = \boldsymbol{u} + O(h^2), \qquad \text{and} \qquad \nabla_h \cdot \boldsymbol{u}_h = 0.$$

A similar case will be in three dimensions. ∎

Replacing the velocity $\boldsymbol{u}$ with $\boldsymbol{u}_h$ in (4.14), (4.15), (4.16), (4.17), (5.1), (5.2), (5.3), the order of the truncation error will not be affected. We still abbreviate $\boldsymbol{u}_h$ as $\boldsymbol{u}$ for the simplicity in what follows.

5.2. Analysis of the error of $\boldsymbol{u}$. Define

$$q^n = p^n - \bar{p}^n, \quad E_\tau^{kl,n} = \tau^{kl,n} - \bar{\tau}^{kl,n}, \quad E^{l,m,n} = \tilde{Q}^{l,m,n} - \bar{Q}^{l,m,n};$$

then by subtracting (2.2), (2.3), (2.4) and (5.1), (5.2), (5.3), we have

$$\frac{e^{1,n+1}-e^{1,n}}{\delta t} + e^{1,n}\bar{D}_x u^{1,n} + \bar{u}^{1,n}\bar{D}_x e^{1,n} + E_xE_y e^{2,n}\bar{D}_y u^{1,n} + E_xE_y \bar{u}^{2,n}\bar{D}_y e^{1,n}$$
$$+ D_x q^{n+1} = \Delta_h e^{1,n+1} + \bar{D}_x E_y E_\tau^{11,n} + D_y E_\tau^{21,n} + O(\delta t + h^2), \tag{5.5}$$

$$\frac{e^{2,n+1}-e^{2,n}}{\delta t} + E_xE_y e^{1,n}\bar{D}_x u^{2,n} + E_xE_y \bar{u}^{1,n}\bar{D}_x e^{2,n} + e^{2,n}\bar{D}_y u^{2,n} + \bar{u}^{2,n}\bar{D}_y e^{2,n}$$
$$+ D_y q^{n+1} = \Delta_h e^{2,n+1} + D_x E_\tau^{12,n} + \bar{D}_y E_x E_\tau^{22,n} + O(\delta t + h^2), \tag{5.6}$$

$$D_x e^{1,n+1} + D_y e^{2,n+1} = 0. \tag{5.7}$$

In order to apply the inverse inequality trick to control the L_h^∞ norm of related variables, we define the "blow-up" time

$$n_{\max} := \max_n \left\{ 0 \le n \le \frac{T}{\delta t} \;\middle|\; \|\langle |\bar{\boldsymbol{Q}}^{\cdot,\cdot,k}|^2\rangle_N\|_{L_h^\infty} \preceq 2(B_1+1), \right. \tag{5.8}$$

$$\|\bar{\boldsymbol{u}}^{\cdot,k}\|_{L_h^\infty} \preceq \|\boldsymbol{u}\|_{C^0(D\times[0,T])}+1,\ \delta t \sum_{l=0}^{k} \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,l}\|_{L_h^\infty}^2 \preceq T\|\boldsymbol{u}\|_{C^1(D\times[0,T])}^2+1,$$

$$\left. \delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,k}\|_{L_h^\infty} \preceq \frac{1}{4}, \delta t \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,k}\|_{L_h^\infty}^2 \preceq \frac{1}{8}, 0 \le k \le n \right\},$$

where B_1 is defined in Lemma 4.9.

LEMMA 5.2. *For all $n \le n_{\max}$ we have the estimate*

$$\frac{1}{2\delta t}\Big(\|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 - \|\boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2 + \|\boldsymbol{e}^{\cdot,n+1}-\boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2\Big) + \frac{1}{2}\|\nabla_h \boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 \tag{5.9}$$

$$\preceq C_1\|\boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2 + C_2\|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 + \frac{1}{8}\|\nabla_h \boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2 + \frac{1}{2}\|E_\tau^{\cdot,n}\|_{L_h^2}^2 + O(\delta t^2+h^4),$$

where $C_1 > 0$ depends on $\|\boldsymbol{u}\|_{C^1(D\times[0,T])}$, and $C_2 > 0$ depends on $\|\boldsymbol{u}\|_{C^0(D\times[0,T])}$. Here

$$\|E_\tau^{\cdot,n}\|_{L_h^2}^2 := \sum_{k,\ l=1}^{2} \|E_\tau^{kl,n}\|_{L_h^2}^2,$$

and

$$\|E_\tau^{kl,n}\|_{L_h^2}^2 \preceq \frac{2d}{N^{1-\epsilon}} + C_3(\delta t^2+h^4) + \Big(\Big\|\big\langle|\tilde{Q}^{k,\cdot,n}|^2\big\rangle_N\Big\|_{L_h^\infty} \tag{5.10}$$

$$+ \Big\|\big\langle|\bar{Q}^{l,\cdot,n}|^2\big\rangle_N\Big\|_{L_h^\infty}\Big)\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n}\|_{L_h^2}^2\Big\rangle_N,$$

where C_3 is positive and depends on $\|\boldsymbol{u}\|_{C^4(D\times[0,T])}$.

Proof. Taking inner product to both sides of (5.5) and (5.6) with $e^{l,n+1}$ and taking summation, we will consider term by term for (5.5).

The first term at the left-hand side of the equation can be estimated as

$$\sum_{ij,l}(e^{l,n+1}-e^{l,n})\cdot e^{l,n+1} = \frac{1}{2}\Big(\|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 - \|\boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2 + \|\boldsymbol{e}^{\cdot,n+1}-\boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2\Big).$$

The second to fifth terms can be estimated by Cauchy's inequality, so they can be bounded by

$$(1+8\|\bar{\boldsymbol{u}}^n\|_{L_h^\infty}^2)\|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 + \|\bar{\nabla}_h \boldsymbol{u}^n\|_{L_h^\infty}^2 \|\boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2 + \frac{1}{8}\|\nabla_h \boldsymbol{e}^{\cdot,n}\|_{L_h^2}^2.$$

The sixth term at the left-hand side of the equation will be 0 by summation by parts and (5.7). The first term at the right-hand side can be estimated by (3.3). The second and third terms can be estimated by summation by parts and Cauchy's inequality, so they can be bounded by

$$\frac{1}{2}\|\nabla_h \boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 + \frac{1}{2}\|E_\tau^{\cdot,n}\|_{L_h^2}^2.$$

Finally, by using

$$\|\bar{\nabla}_h \boldsymbol{u}^n\|_{L_h^\infty}^2 \le \|\boldsymbol{u}\|_{C^1(D\times[0,T])}^2$$

and the condition $\|\bar{\boldsymbol{u}}^n\|_{L_h^\infty}^2 \le \|\boldsymbol{u}\|_{C^0(D\times[0,T])} + 1$, we have the inequality (5.9).

For the error of polymeric stress we have

$$\begin{aligned} \|E^{kl,n}\|_{L_h^2}^2 &= \|\langle Q^{k,n}Q^{l,n}\rangle - \langle \bar{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\rangle_N\|_{L_h^2}^2 \\ &\le \|\langle Q^{k,n}Q^{l,n}\rangle - \langle \tilde{Q}^{k,n}\tilde{Q}^{l,n}\rangle\|_{L_h^2}^2 + \|\langle \tilde{Q}^{k,n}\tilde{Q}^{l,n}\rangle - \langle \tilde{Q}^{k,\cdot,n}\tilde{Q}^{l,\cdot,n}\rangle_N\|_{L_h^2}^2 \\ &+ \|\langle \tilde{Q}^{k,\cdot,n}\tilde{Q}^{l,\cdot,n}\rangle_N - \langle \bar{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\rangle_N\|_{L_h^2}^2 \\ &:= P_1 + P_2 + P_3, \end{aligned} \tag{5.11}$$

where P_1 accounts for the discretization error with $\boldsymbol{u}$ exact, P_2 accounts for Monte Carlo discretization, and P_3 accounts for the discretization error with $\boldsymbol{u}$ replaced by $\bar{\boldsymbol{u}}$.

We have

$$P_1 \le \|\mathbb{E}(\boldsymbol{Q}^n)^2\|_{L_h^\infty}\mathbb{E}\|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2 + \|\mathbb{E}(\tilde{\boldsymbol{Q}}^n)^2\|_{L_h^\infty}\mathbb{E}\|\tilde{\boldsymbol{E}}^n\|_{L_h^2}^2 \le C(\delta t^2 + h^4) \tag{5.12}$$

by Lemmas 4.4 and 4.9,

$$P_2 \preceq \frac{2d}{N^{1-\epsilon}} \tag{5.13}$$

by Lemma 4.10, and

$$\begin{aligned} &|\langle \tilde{Q}^{k,\cdot,n}\tilde{Q}^{l,\cdot,n}\rangle_N - \langle \bar{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\rangle_N| \\ \le\ &|\langle \tilde{Q}^{k,\cdot,n}\tilde{Q}^{l,\cdot,n}\rangle_N - \langle \tilde{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\rangle_N| + |\langle \tilde{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\rangle_N - \langle \bar{Q}^{k,\cdot,n}\bar{Q}^{l,\cdot,n}\rangle_N| \\ \le\ &\left\langle |\tilde{Q}^{k,\cdot,n}||\bar{E}^{l,\cdot,n}|\right\rangle_N + \left\langle |\bar{E}^{k,\cdot,n}||\bar{Q}^{l,\cdot,n}|\right\rangle_N; \end{aligned}$$

thus by Cauchy's inequality we have

$$P_3 \le \left(\left\|\left\langle|\tilde{Q}^{k,\cdot,n}|^2\right\rangle_N\right\|_{L_h^\infty} + \left\|\left\langle|\bar{Q}^{l,\cdot,n}|^2\right\rangle_N\right\|_{L_h^\infty}\right)\left\langle\|\boldsymbol{E}^{\cdot,\cdot,n}\|_{L_h^2}^2\right\rangle_N. \tag{5.14}$$

The proof of Lemma 5.2 is complete. ∎

5.3. Analysis of the error of $\boldsymbol{Q}$. Subtracting the discretized equation (2.5), (2.6) of $\bar{Q}$ and (4.25), (4.26) of $\tilde{Q}$, we have

$$\begin{aligned} E^{1,m,n+1} = E^{1,m,n} &- \Big(E_y e^{1,n}\bar{D}_x\tilde{Q}^{1,m,n+1} + E_y\bar{u}^{1,n}\bar{D}_x E^{1,m,n+1} \\ &+ E_x e^{2,n}\bar{D}_y\tilde{Q}^{1,m,n+1} + E_x\bar{u}^{2,n}\bar{D}_y E^{1,m,n+1}\Big)\delta t + \Big(\bar{D}_x E_y e^{1,n}\tilde{Q}^{1,m,n} + \bar{D}_x E_y\bar{u}^{1,n}E^{1,m,n} \\ &+ D_x e^{1,n}\tilde{Q}^{2,m,n} + D_x\bar{u}^{1,n}E^{2,m,n}\Big)\delta t - E^{1,m,n}\delta t, \end{aligned} \tag{5.15}$$

$$\begin{aligned} E^{2,m,n+1} = E^{2,m,n} &- \Big(E_y e^{1,n}\bar{D}_x\tilde{Q}^{2,m,n+1} + E_y\bar{u}^{1,n}\bar{D}_x E^{2,m,n+1} \\ &+ E_x e^{2,n}\bar{D}_y\tilde{Q}^{2,m,n+1} + E_x\bar{u}^{2,n}\bar{D}_y E^{2,m,n+1}\Big)\delta t + \Big(D_x e^{2,n}\tilde{Q}^{1,m,n} + D_x\bar{u}^{2,n}E^{1,m,n} \\ &+ \bar{D}_y E_x e^{2,n}\tilde{Q}^{2,m,n} + \bar{D}_y E_x\bar{u}^{2,n}E^{2,m,n}\Big)\delta t - E^{2,m,n}\delta t. \end{aligned} \tag{5.16}$$

Timing both sides with $E^{l,m,n+1}$ and taking summation on all grid points, we will obtain the discrete norm estimate. Because the whole procedure is tedious, we will pay attention to the convection term at first. For simplicity we define $\boldsymbol{P}^{m,n+1}$ for the convective terms

$$\boldsymbol{P}^{m,n+1} = \begin{pmatrix} E_y\bar{u}^{1,n}\bar{D}_x E^{1,m,n+1} + E_x\bar{u}^{2,n}\bar{D}_y E^{1,m,n+1} \\ E_y\bar{u}^{1,n}\bar{D}_x E^{2,m,n+1} + E_x\bar{u}^{2,n}\bar{D}_y E^{2,m,n+1} \end{pmatrix}$$

and $P^{l,m,n+1}_{ij}$ for its lth component at corresponding node $(i+\frac{1}{2}, j)$ or $(i, j+\frac{1}{2})$; then we have the following lemma.

LEMMA 5.3. *Considering the discretization effect of convection term*

$$\begin{aligned} P^{l,m,n+1}_{i,j} \cdot E^{l,m,n+1}_{i,j} &= E^{l,m,n+1}_{i,j} \cdot E_y\bar{u}^{1,n}_{i,j} \cdot (E^{l,m,n+1}_{i+1,j} - E^{l,m,n+1}_{i-1,j})/2h \\ &\quad + E^{l,m,n+1}_{i,j} \cdot E_x\bar{u}^{2,n}_{i,j} \cdot (E^{l,m,n+1}_{i,j+1} - E^{l,m,n+1}_{i,j-1})/2h, \end{aligned}$$

we have the estimate

$$\|\boldsymbol{P}^{m,n+1} \cdot \boldsymbol{E}^{l,m,n+1}\|_{L^1_h} \leq \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h} \|\boldsymbol{E}^{m,n+1}\|^2_{L^2_h}. \tag{5.17}$$

Proof. Summation by parts shows that

$$\begin{aligned} \sum_{i,j} P^{l,m,n+1}_{i,j} \cdot E^{l,m,n+1}_{i,j} &= \sum_{i,j} (E_y\bar{u}^{1,n}_{i-1,j} - E_y\bar{u}^{1,n}_{i,j}) E^{l,m,n+1}_{i,j} E^{l,m,n+1}_{i-1,j}/2h \\ &\quad + \sum_{i,j} (E_x\bar{u}^{2,n}_{i,j-1} - E_x\bar{u}^{2,n}_{i,j}) E^{l,m,n+1}_{i,j} E^{l,m,n+1}_{i,j-1}/2h. \end{aligned}$$

Then the standard inequality

$$E^{l,m,n+1}_{i,j} E^{l,m,n+1}_{i-1,j} \leq \Big((E^{l,m,n+1}_{i,j})^2 + (E^{l,m,n+1}_{i-1,j})^2 \Big) \Big/ 2$$

and summation with respect to l give the estimate (5.17). ∎

LEMMA 5.4. *For all $n \leq n_{\max}$ and $\delta t \leq \frac{1}{2}$, we have the following estimate for the error $\boldsymbol{E}$:*

$$\begin{aligned} \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h} \Big\rangle_N &\preceq \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h} \Big\rangle_N + C_1\delta t \|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + \frac{\delta t}{2} \|\nabla_h \boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} \\ &\quad + C_2\delta t \big(1 + \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h}\big) \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h} \Big\rangle_N, \end{aligned} \tag{5.18}$$

where C_1 depends on B_2, and C_2 depends on B_1.

Proof. For simplicity we define

$$\boldsymbol{G}^{m,n+1} = \begin{pmatrix} E_y e^{1,n}\bar{D}_x\tilde{Q}^{1,m,n+1} + E_x e^{2,n}\bar{D}_y\tilde{Q}^{1,m,n+1} \\ E_y e^{1,n}\bar{D}_x\tilde{Q}^{2,m,n+1} + E_x e^{2,n}\bar{D}_y\tilde{Q}^{2,m,n+1} \end{pmatrix}$$

for the rest of the convective terms and

$$\boldsymbol{H}^{m,n+1} = \begin{pmatrix} \bar{D}_x E_y e^{1,n}\tilde{Q}^{1,m,n} + \bar{D}_x E_y\bar{u}^{1,n} E^{1,m,n} + D_x e^{1,n}\tilde{Q}^{2,m,n} + D_x\bar{u}^{1,n} E^{2,m,n} \\ D_x e^{2,n}\tilde{Q}^{1,m,n} + D_x\bar{u}^{2,n} E^{1,m,n} + \bar{D}_y E_x e^{2,n}\tilde{Q}^{2,m,n} + \bar{D}_y E_x\bar{u}^{2,n} E^{2,m,n} \end{pmatrix}$$

for the terms related to $\kappa \cdot \boldsymbol{Q}$. W define $G^{l,m,n+1}_{ij}, H^{l,m,n+1}_{ij}$ for their lth components at corresponding node $(i+\frac{1}{2}, j)$ or $(i, j+\frac{1}{2})$. Timing both sides of (5.15) and (5.16) with $E^{l,m,n+1}$, we will consider each term with the following abbreviations:

$$E^{l,m,n+1} - E^{l,m,n} = \delta t (P^{l,m,n+1} + G^{l,m,n+1} + H^{l,m,n+1} - E^{l,m,n}). \tag{5.19}$$

We can obtain the following estimate after minor manipulations:

$$\frac{1}{2}\Big((E^{l,m,n+1})^2-(E^{l,m,n})^2+(E^{l,m,n+1}-E^{l,m,n})^2\Big)\le\frac{\delta t}{2}(G^{l,m,n+1})^2 + \frac{\delta t}{32B_1}(H^{l,m,n+1})^2+\frac{\delta t}{2}(E^{l,m,n})^2+\delta t(1+8B_1)(E^{l,m,n+1})^2 + \delta t P^{l,m,n+1}\cdot E^{l,m,n+1}.$$

We have

$$h^d\left\langle\sum_{ij,l}(G^{l,m,n+1}_{i,j})^2\right\rangle_N\le\|\langle|\bar\nabla_h\tilde Q^{l,\cdot,n+1}|^2\rangle_N\|_{L^\infty_h}\|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h}\preceq B_2\|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} \tag{5.20}$$

by Lemma 4.9. For the $\kappa\cdot\boldsymbol{Q}$ term we have

$$\begin{aligned}h^d\left\langle\sum_{ij,l}(H^{l,m,n+1}_{i,j})^2\right\rangle_N&\le 4\|\langle|\tilde Q^{l,\cdot,n}|^2\rangle_N\|_{L^\infty_h}\|\nabla_h\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h}+8\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h}\|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\\&\preceq 4B_1\|\nabla_h\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h}+8\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h}\|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\end{aligned} \tag{5.21}$$

by Lemma 4.9. The term $\delta t P^{l,m,n+1}\cdot E^{l,m,n+1}$ can be estimated by Lemma 5.3.

Combining all the estimates above we have

$$\begin{aligned}&\frac{1}{2}\Big(\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h}\Big\rangle_N-\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N+\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n+1}-\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N\Big)\\&\preceq\frac{B_2\delta t}{2}\|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h}+\frac{\delta t}{8}\|\nabla_h\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h}+\delta t(\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h}+1+8B_1)\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h}\Big\rangle_N\\&\quad+\delta t\bigg(\frac{1}{2}+\frac{1}{4B_1}\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h}\bigg)\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N.\end{aligned} \tag{5.22}$$

Moving the $\boldsymbol{E}^{\cdot,\cdot,n+1}$ term to the left-hand side and dividing the coefficient

$$1-2\delta t(\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h}+1+8B_1)$$

to both sides, the only term causing trouble is $1-2\delta t\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h}$. By condition $n\le n_{\max}$ we have

$$1\le(1-2\delta t\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h})^{-1}\preceq 1+\delta t(2\|\nabla_h\bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h}+1)\preceq 2. \tag{5.23}$$

Some simple manipulations give inequality (5.18). ∎

LEMMA 5.5. *For all $n\le n_{\max}$ we have*

$$\|\boldsymbol{e}^{\cdot,n+1}\|^2_{L^2_h}+\Big\langle\|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h}\Big\rangle_N\preceq C_3\bigg(\delta t^2+h^4+\frac{1}{N^{1-\epsilon}}\bigg) \tag{5.24}$$

and

$$\sum_{k=0}^{n+1}\|\nabla_h\boldsymbol{e}^{\cdot,k}\|^2_{L^2_h}\preceq C_3\bigg(\delta t^2+h^4+\frac{1}{N^{1-\epsilon}}\bigg), \tag{5.25}$$

where C_3 depends on $\|\boldsymbol{u}\|_{C^4(D\times[0,T])}$, C_1, and C_2 in Lemma 5.4.

Proof. From Lemma 5.2, we have

$$\|\boldsymbol{e}^{\cdot,n+1}\|^2_{L^2_h} + \delta t\|\nabla_h \boldsymbol{e}^{\cdot,n+1}\|^2_{L^2_h} \le (1+C\delta t)\|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + \frac{\delta t}{2}\|\nabla_h \boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + \delta t\|E^{\cdot,n}_\tau\|^2_{L^2_h} + C(\delta t^2+h^4)\delta t. \tag{5.26}$$

From Lemma 5.4, we have

$$\Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h}\Big\rangle_N \preceq \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N + C\delta t\|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + \frac{\delta t}{2}\|\nabla_h \boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + C\delta t\big(1+\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h}\big)\Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N. \tag{5.27}$$

From the error estimate of stress (5.10) and assumption, we have

$$\|E^{\cdot,n}_\tau\|^2_{L^2_h} \preceq \frac{2d}{N^{1-\epsilon}} + C(\delta t^2+h^4) + C\Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N. \tag{5.28}$$

Summing up all the inequalities, we obtain

$$\begin{aligned}\|\boldsymbol{e}^{\cdot,n+1}\|^2_{L^2_h} + \delta t\|\nabla_h \boldsymbol{e}^{\cdot,n+1}\|^2_{L^2_h} + \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h}\Big\rangle_N &\preceq \Big(1+C\delta t(\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h}+1)\Big)\\ \cdot\left(\|\boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + \delta t\|\nabla_h \boldsymbol{e}^{\cdot,n}\|^2_{L^2_h} + \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n}\|^2_{L^2_h}\Big\rangle_N\right) &+ C\left(\delta t^2 + h^4 + \frac{1}{N^{1-\epsilon}}\right)\delta t.\end{aligned} \tag{5.29}$$

By the discrete Gronwall inequality and the condition $n \le n_{\max}$, we have

$$\|\boldsymbol{e}^{\cdot,n+1}\|^2_{L^2_h} + \Big\langle \|\boldsymbol{E}^{\cdot,\cdot,n+1}\|^2_{L^2_h}\Big\rangle_N \preceq C\left(\delta t^2 + h^4 + \frac{1}{N^{1-\epsilon}}\right)$$

and

$$\sum_{k=0}^{n+1}\|\nabla_h \boldsymbol{e}^{\cdot,k}\|^2_{L^2_\tau L^2_h} \preceq C\left(\delta t^2 + h^4 + \frac{1}{N^{1-\epsilon}}\right).$$

The proof is complete. ◻

5.4. Inverse estimate. The final job is to prove $n_{\max} = \frac{T}{\delta t}$ by a continuation technique with inverse inequality. We have the following lemma.

LEMMA 5.6. *With the error estimate in Lemma* 5.5*, the following inequalities hold for all* $0 \le n \le \frac{T}{\delta t}$*; i.e., we have* $n_{\max} = \frac{T}{\delta t}$*:*

$$\left\|\big\langle|\bar{\boldsymbol{Q}}^{\cdot,\cdot,n}|^2\big\rangle_N\right\|_{L^\infty_h} \preceq 2\,(B_1+1)\,, \quad \|\bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h} \preceq \|\boldsymbol{u}\|_{C^0(D\times[0,T])} + 1,$$

$$\delta t\sum_n \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h} \preceq T\|\boldsymbol{u}\|^2_{C^1(D\times[0,T])} + 1,\ \ \delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|_{L^\infty_h} \preceq \frac{1}{4},\ \ \delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n}\|^2_{L^\infty_h} \preceq \frac{1}{8}.$$

Proof. We have for all $n \le n_{\max}$

$$\begin{aligned}\left\|\big\langle|\bar{\boldsymbol{Q}}^{\cdot,\cdot,n+1}|^2\big\rangle_N\right\|_{L^\infty_h} &\le \left(\left\|\big\langle|\bar{\boldsymbol{Q}}^{\cdot,\cdot,n+1} - \tilde{\boldsymbol{Q}}^{\cdot,\cdot,n+1}|^2\big\rangle_N\right\|_{L^\infty_h} + \left\|\big\langle|\tilde{\boldsymbol{Q}}^{\cdot,\cdot,n+1}|^2\big\rangle_N\right\|_{L^\infty_h}\right)\cdot 2\\ &\preceq \left(\left\|\big\langle|\boldsymbol{E}^{\cdot,\cdot,n+1}|^2\big\rangle_N\right\|_{L^\infty_h} + B_1\right)\cdot 2,\end{aligned}$$

and

$$\left\|\left\langle|\boldsymbol{E}^{\cdot,\cdot,n+1}|^2\right\rangle_N\right\|_{L_h^\infty} \le \left\langle\|\boldsymbol{E}^{\cdot,\cdot,n+1}\|_{L_h^\infty}^2\right\rangle_N \le \left\langle\left(h^{-\frac{d}{2}}\|\boldsymbol{E}^{\cdot,\cdot,n+1}\|_{L_h^2}\right)^2\right\rangle_N$$
$$= h^{-d}\left\langle\|\boldsymbol{E}^{\cdot,\cdot,n+1}\|_{L_h^2}^2\right\rangle_N \preceq h^{-d}\left(\frac{1}{N^{1-\epsilon}} + \delta t^2 + h^4\right).$$

If we choose $\delta t = h^2$, $N = h^{-\alpha}$, in order to ensure convergence, we need

$$h^{-d}(h^{\alpha(1-\epsilon)} + h^4) \longrightarrow 0 \quad \text{as} \quad h \to 0,$$

i.e., $\alpha(1-\epsilon) > d$, which is equivalent to

$$\alpha > d.$$

That is the condition we need in Theorem 2.2.

For $\|\bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty}$ we have

$$\|\bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty} \le \|\bar{\boldsymbol{u}}^{\cdot,n+1} - \boldsymbol{u}^{\cdot,n+1}\|_{L_h^\infty} + \|\boldsymbol{u}^{\cdot,n+1}\|_{L_h^\infty} \le \|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^\infty} + \|\boldsymbol{u}\|_{C^0(D\times[0,T])},$$

and

$$\|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^\infty} \le h^{-\frac{d}{2}}\|\boldsymbol{e}^{\cdot,n+1}\|_{L_h^2} \preceq h^{-\frac{d}{2}}(N^{-\frac{1-\epsilon}{2}} + \delta t + h^2) \longrightarrow 0$$

as $h \to 0$.

For $\delta t \sum_{k=0}^{n+1} \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,k}\|_{L_h^\infty}^2$ we have

$$\delta t \sum_{k=0}^{n+1} \|\nabla_h \bar{\boldsymbol{u}}^{\cdot,k}\|_{L_h^\infty}^2 \le \delta t \sum_{k=0}^{n+1} \|\nabla_h \boldsymbol{e}^{\cdot,k}\|_{L_h^\infty}^2 + \delta t \sum_{k=0}^{n+1} \|\nabla_h \boldsymbol{u}^{\cdot,k}\|_{L_h^\infty}^2$$
$$\le h^{-d}\delta t \sum_{k=0}^{n+1} \|\nabla_h \boldsymbol{e}^{\cdot,k}\|_{L_h^2}^2 + T\|\boldsymbol{u}\|_{C^1(D\times[0,T])}^2.$$

In a similar argument as that for $\|\bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty}$, we have

$$h^{-d}\delta t \sum_{k=0}^{n+1} \|\nabla_h \boldsymbol{e}^{\cdot,k}\|_{L_h^2}^2 \longrightarrow 0$$

as $h \to 0$.

For $\delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty}^2$, we have

$$\delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty}^2 \le 2\delta t\|\nabla_h \boldsymbol{e}^{\cdot,n+1}\|_{L_h^\infty}^2 + 2\delta t\|\nabla_h \boldsymbol{u}^{\cdot,n+1}\|_{L_h^\infty}^2$$
$$\le 2h^{-d}\delta t\|\nabla_h \boldsymbol{e}^{\cdot,n+1}\|_{L_h^2}^2 + 2\delta t\|\boldsymbol{u}\|_{C^1(D\times[0,T])}.$$

In a similar argument as above, it is quite clear $\delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty}^2 \to 0$. The reason for $\delta t\|\nabla_h \bar{\boldsymbol{u}}^{\cdot,n+1}\|_{L_h^\infty}$ is the same.

By continuation technique, we know $n_{\max} = \frac{T}{\delta t}$ in (5.8); thus the proof is ended. ∎

6. Conclusion. The convergence analysis for the BCF method with MAC scheme in dimension 2 or 3 for the Hookean dumbbell model is performed in this paper. It is shown that if the number of configuration fields N, the space stepsize h and the time stepsize δt are chosen appropriately, the convergence of second order in space and first order in time is obtained after excluding a set of exponentially small probability. The inverse inequality trick is the key step for the numerical analysis of the coupled system. The explicit large deviation estimates for $(\tilde{Q}^k)^2$ and $\tilde{Q}^k\tilde{Q}^l$ for the empirical polymeric stress are the central issue. Further investigations on the convergence analysis for nonlinear dumbbell models are needed in the future work.

Appendix A. Proof of Lemma 4.7. The proof is obtained by a direct computation. First, consider the logarithmic moment generating function

$$\Lambda(\lambda) = \ln \mathbb{E} e^{\lambda \tilde{Q}^2} = \ln \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} e^{\lambda x^2} e^{-\frac{x^2}{2\tilde{\sigma}^2}} dx = -\frac{1}{2}\ln(1-2\lambda\tilde{\sigma}^2).$$

Then from $\frac{\partial}{\partial\lambda}[\lambda x - \Lambda(\lambda)] = 0$, we have

$$x = \frac{\tilde{\sigma}^2}{1-2\lambda\tilde{\sigma}^2}.$$

The existence of exponential moment needs

$$1 - 2\lambda\tilde{\sigma}^2 > 0, \quad \text{i.e.,} \quad \lambda < \frac{1}{2\tilde{\sigma}^2}.$$

But when $\lambda \in (-\infty, \frac{1}{2\tilde{\sigma}^2})$, $x \in (0, +\infty)$, and $\Lambda'(\lambda) > 0$, then there exists only one

$$\lambda(x) = \frac{1}{2\tilde{\sigma}^2} - \frac{1}{2x}$$

for arbitrary $x \in (0, +\infty)$. Thus

$$I(x) = \frac{x}{2\tilde{\sigma}^2} - \frac{1}{2} + \frac{1}{2}\ln\frac{\tilde{\sigma}^2}{x}.$$

The proof is complete.

Appendix B. Proof of Lemma 4.8. The proof is obtained by a direct computation. First, consider the logarithmic moment generating function

$$\begin{aligned}\Lambda(\lambda) = \ln \mathbb{E} e^{\lambda \tilde{Q}_1\tilde{Q}_2} &= \ln \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi \det(\tilde{\Sigma})^{\frac{1}{2}}} e^{\lambda xy} e^{-\frac{1}{2}X^T\tilde{\Sigma}^{-1}X} dxdy \\ &= \ln \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi \det(\tilde{\Sigma})^{\frac{1}{2}}} e^{-\frac{1}{2}X^T(\tilde{\Sigma}^{-1}+\lambda S)X} dxdy \\ &= -\frac{1}{2}\ln(1-2\lambda\tilde{\sigma}_{12} - \lambda^2\det(\tilde{\Sigma})),\end{aligned}$$

where $X = (x,y)^T$, $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Then from $\frac{\partial}{\partial\lambda}[\lambda x - \Lambda(\lambda)] = 0$, we have

$$x = \frac{\tilde{\sigma}_{12} + \lambda\det(\tilde{\Sigma})}{1-2\lambda\tilde{\sigma}_{12} - \lambda^2\det(\tilde{\Sigma})}.$$

The existence of exponential moment needs

$$1 - 2\lambda\tilde{\sigma}_{12} - \lambda^2\det(\tilde{\Sigma}) > 0, \quad \text{i.e.,} \quad \lambda_- < \lambda < \lambda_+,$$

where $\lambda_\pm = \frac{-\tilde{\sigma}_{12} \pm \sqrt{\tilde{\sigma}_{12}^2 + \det(\tilde{\Sigma})}}{\det(\tilde{\Sigma})}$. But when $\lambda \in (\lambda_-, \lambda_+)$, $x \in (-\infty, +\infty)$, and $\Lambda'(\lambda) > 0$, then there exists only one $\lambda(x)$ for arbitrary $x \in R$.

Appendix C. Estimate for remainders in Lemma 4.4. Let us first consider the expressions and estimates for remainder terms $R^{l,n}$ in (4.16) and (4.17). We will consider only $R^{2,n}$ here; a similar case will be for $R^{1,n}$. Finally, we will obtain the decompostion

$$\boldsymbol{R}^n = \boldsymbol{R}_1^n + \boldsymbol{R}_2^n.$$

$\boldsymbol{R}_i^n$ satisfy the following estimates:

$$\mathbb{E}\|\boldsymbol{R}_1^n\|_{L_h^2}^2 \le C_1 \delta t^3, \qquad \mathbb{E}\|\boldsymbol{R}_2^n\|_{L_h^2}^2 \le C_2 \delta t^2 (h^4 + \delta t^2),$$

where C_1 and C_2 are positive constants depending on $\|\boldsymbol{u}\|_{C^4(D\times[0,T])}$, and $\boldsymbol{R}_1^n$ is the martingale part which satisfies $\mathbb{E}(\boldsymbol{R}_1^n \cdot \tilde{\boldsymbol{E}}^n) = 0$. The concrete expressions of $\boldsymbol{R}_1^n$ are composed of $P_{6,3}^B$, $P_{8,3}^B$, and $P_{9,3}^B$ in (C.1), (C.2), and (C.3). Now we have

$$R^{2,n} = \frac{1}{4h^2} \cdot (-I + II + III + IV + V)$$

and

$$\begin{aligned}
I &= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \Big(Q^2(x, y, t_{n+1}) - Q^2(x_i, y_j, t_{n+1}) \Big) dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \left(\int_{x_i}^{x} Q_x^2(\tilde{x}, y, t_{n+1}) d\tilde{x} + \int_{y_j}^{y} Q_y^2(x_i, \tilde{y}, t_{n+1}) d\tilde{y} \right) dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \left(\int_{x_i}^{x} \int_{x_i}^{\tilde{x}} Q_{xx}^2(\tilde{\tilde{x}}, y, t_{n+1}) d\tilde{\tilde{x}} d\tilde{x} + \int_{y_j}^{y} \int_{y_j}^{\tilde{y}} Q_{yy}^2(x_i, \tilde{\tilde{y}}, t_{n+1}) d\tilde{\tilde{y}} d\tilde{y} \right) dy dx,
\end{aligned}$$

$$\begin{aligned}
II &= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \Big(Q^2(x, y, t_n) - Q^2(x_i, y_j, t_n) \Big) dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \left(\int_{x_i}^{x} Q_x^2(\tilde{x}, y, t_n) d\tilde{x} + \int_{y_j}^{y} Q_y^2(x_i, \tilde{y}, t_n) d\tilde{y} \right) dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \left(\int_{x_i}^{x} \int_{x_i}^{\tilde{x}} Q_{xx}^2(\tilde{\tilde{x}}, y, t_n) d\tilde{\tilde{x}} d\tilde{x} + \int_{y_j}^{y} \int_{y_j}^{\tilde{y}} Q_{yy}^2(x_i, \tilde{\tilde{y}}, t_n) d\tilde{\tilde{y}} d\tilde{y} \right) dy dx.
\end{aligned}$$

So we have

$$\begin{aligned}
I - II = \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \Bigg(& \int_{x_i}^{x} \int_{x_i}^{\tilde{x}} \int_{t_n}^{t_{n+1}} Q_{xxs}^2(\tilde{\tilde{x}}, y, s) ds d\tilde{\tilde{x}} d\tilde{x} \\
& + \int_{y_j}^{y} \int_{y_j}^{\tilde{y}} \int_{t_n}^{t_{n+1}} Q_{yys}^2(x_i, \tilde{\tilde{y}}, s) ds d\tilde{\tilde{y}} d\tilde{y} \Bigg) dy dx.
\end{aligned}$$

From this formula and Lemma 4.3, we obtain $\mathbb{E}|I - II| \le Ch^4 \delta t$.

$$\begin{aligned}
III = \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(& -\boldsymbol{u} \cdot \nabla Q^2 + E_y u^{1,n} \bar{D}_x Q^{2,n+1} \\
& + E_x u^{2,n} \bar{D}_y Q^{2,n+1} \Big) dt dy dx = III_1 + III_2.
\end{aligned}$$

Here

$$
\begin{aligned}
III_1 = &\int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(-u^1 Q_x^2 + E_y u^{1,n} Q_x^2 - E_y u^{1,n} Q_x^2 \\
&+ E_y u^{1,n} \bar{D}_x Q^{2,n+1} \Big) dt dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big((-u^1 + E_y u^{1,n}) Q_x^2 - E_y u^{1,n} (Q_x^2 - \bar{D}_x Q^2) \Big) dt dy dx \\
&= P_1 - P_2.
\end{aligned}
$$

From this formula, we obtain $\mathbb{E}|P_1| \le Ch^4 \delta t$.

For P_2, we have the following estimate:

$$
\begin{aligned}
P_2 &= E_y u^{1,n} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(Q_x^2 - \bar{D}_x Q^{2,n+1} \Big) dt dy dx \\
&= E_y u^{1,n} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\Big(Q^2(x_{i+1}, y, t) - Q^2(x_{i+1}, y_j, t_{n+1}) \Big) \\
&- \Big(Q^2(x_{i-1}, y, t) - Q^2(x_{i-1}, y_j, t_{n+1}) \Big) \Big) dt dy \\
&= E_y u^{1,n} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{y_j}^{y} \int_{x_{i-1}}^{x_{i+1}} Q_{xy}^2(x, \tilde{y}, t) dx d\tilde{y} \\
&+ \int_t^{t_{n+1}} \int_{x_{i-1}}^{x_{i+1}} Q_{xs}^2(x, y_j, s) dx ds \Big) dt dy \\
&= E_y u^{1,n} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{y_j}^{y} \int_{x_{i-1}}^{x_{i+1}} \int_{y}^{y_j} Q_{xyy}^2(x, \tilde{\tilde{y}}, t) d\tilde{\tilde{y}} dx d\tilde{y} \\
&+ \int_t^{t_{n+1}} \int_{x_{i-1}}^{x_{i+1}} Q_{xs}^2(x, y_j, s) dx ds \Big) dt dy \\
&= P_{2,1} + P_{2,2}.
\end{aligned}
$$

From this formula, we obtain $\mathbb{E}|P_{2,1}| \le Ch^4 \delta t$, $\mathbb{E}|P_{2,2}| \le Ch^2 \delta t^2$.

Similarly for III_2, we have

$$
\begin{aligned}
III_2 = &\int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(-u^2 Q_y^2 + E_x u^{2,n} Q_y^2 - E_x u^{2,n} Q_y^2 \\
&+ E_x u^{2,n} \bar{D}_y Q^{2,n+1} \Big) dt dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big((-u^2 + E_x u^{2,n}) Q_y^2 - E_x u^{2,n} (Q_y^2 - \bar{D}_y Q^{2,n+1}) \Big) dt dy dx \\
&= P_3 - P_4.
\end{aligned}
$$

From this formula, we obtain $\mathbb{E}|P_3| \le Ch^4 \delta t$.

For P_4, we have the following estimate:

$$
P_4 = E_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(Q_y^2 - \bar{D}_y Q^{2,n+1} \Big) dt dy dx
$$

$$
\begin{aligned}
&= E_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{t_n}^{t_{n+1}} \Big(\Big(Q^2(x, y_{j+1}, t) - Q^2(x_i, y_{j+1}, t_{n+1}) \Big) \\
&- \Big(Q^2(x, y_{j-1}, t) - Q^2(x_i, y_{j-1}, t_{n+1}) \Big) \Big) dtdx \\
&= E_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{x_i}^{x} \int_{y_{j-1}}^{y_{j+1}} Q^2_{xy}(\tilde{x}, y, t) dy d\tilde{x} \\
&+ \int_{t}^{t_{n+1}} \int_{y_{j-1}}^{y_{j+1}} Q^2_{ys}(x_i, y, s) dxds \Big) dydt \\
&= E_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{x_i}^{x} \int_{y_{j-1}}^{y_{j+1}} \int_{x_i}^{\tilde{x}} Q^2_{xxy}(\tilde{\tilde{x}}, y, t) d\tilde{\tilde{x}} dy d\tilde{x} \\
&+ \int_{t}^{t_{n+1}} \int_{y_{j-1}}^{y_{j+1}} Q^2_{ys}(x_i, y, s) dxds \Big) dydt \\
&= P_{4,1} + P_{4,2}.
\end{aligned}
$$

From this formula, we obtain $\mathbb{E}|P_{4,1}| \le Ch^4 \delta t$, $\mathbb{E}|P_{4,2}| \le Ch^2 \delta t^2$.

$$
\begin{aligned}
IV &= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(u_x^2 Q^1 + u_y^2 Q^2 - D_x u^{2,n} Q^{1,n} - \bar{D}_y E_x u^{2,n} Q^{2,n} \Big) dtdxdy \\
&= IV_1 + IV_2.
\end{aligned}
$$

Here

$$
\begin{aligned}
IV_1 &= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(u_x^2 Q^1 - D_x u^{2,n} Q^1 + D_x u^{2,n} Q^1 - D_x u^{2,n} Q^{1,n} \Big) dtdxdy \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\Big(u_x^2 - D_x u^{2,n} \Big) Q^1 + D_x u^{2,n} \Big(Q^1 - Q^{1,n} \Big) \Big) dtdxdy \\
&= P_5 + P_6.
\end{aligned}
$$

From this formula, we obtain $\mathbb{E}|P_5| \le Ch^4 \delta t$.

For P_6, we have the following estimate:

$$
\begin{aligned}
P_6 &= D_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(Q^1 - Q^{1,n} \Big) dtdxdy \\
&= D_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(Q^1(x, y, t) - Q^1(x_i, y_j, t_n) \Big) dtdxdy \\
&= D_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{x_i}^{x} Q^1_x(\tilde{x}, y, t) d\tilde{x} + \int_{y_j}^{y} Q^1_y(x_i, \tilde{y}, t) d\tilde{y} \\
&+ \int_{t_n}^{t} Q^1_s(x_i, y_j, s) ds \Big) dtdydx \\
&= D_x u^{2,n} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{x_i}^{x} \int_{x_i}^{\tilde{x}} Q^1_{xx}(\tilde{\tilde{x}}, y, t) d\tilde{\tilde{x}} d\tilde{x} \\
&+ \int_{y_j}^{y} \int_{y_j}^{\tilde{y}} Q^1_{yy}(x_i, \tilde{\tilde{y}}, t) \tilde{\tilde{y}} d\tilde{y} + \int_{t_n}^{t} Q^1_s(x_i, y_j, s) ds \Big) dtdydx \\
&= P_{6,1} + P_{6,2} + P_{6,3}.
\end{aligned}
$$

We have

$$\begin{aligned}P_{6,3} &= \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} Q_s^1(x_i,y_j,s)dsdtdydx\\ &= \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(\int_{t_n}^{t}(-\boldsymbol{u}\cdot\nabla\boldsymbol{Q}+\kappa\boldsymbol{Q}-\boldsymbol{Q})^1ds+\int_{t_n}^{t}dW_s^1\Big)dtdydx\\ &= P_{6,3}^A+P_{6,3}^B.\end{aligned}$$

Here $(-\boldsymbol{u}\cdot\nabla\boldsymbol{Q}+\kappa\boldsymbol{Q}-\boldsymbol{Q})^1$ means the first component of the term in the parentheses, and

$$P_{6,3}^B = \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^1 dtdydx \tag{C.1}$$

is one part of $\boldsymbol{R}_1^n$. From this formula and Lemma 4.3, we obtain

$$\mathbb{E}|P_{6,1}|,\ \mathbb{E}|P_{6,2}|\le Ch^4\delta t,\quad \mathbb{E}|P_{6,3}^A|\le Ch^2\delta t^2,\quad \mathbb{E}|P_{6,3}^B|^2\le Ch^4\delta t^3.$$

Similarly for IV_2, we have

$$\begin{aligned}IV_2 &= \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(u_y^2Q^2-\bar{D}_yE_xu^{2,n}Q^2+\bar{D}_yE_xu^{2,n}Q^2\\ &\quad -\bar{D}_yE_xu^{2,n}Q^{2,n}\Big)dtdydx\\ &= \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(\Big(u_y^2-\bar{D}_yE_xu^{2,n}\Big)Q^2+\bar{D}_yE_xu^{2,n}\Big(Q^2-Q^{2,n}\Big)\Big)dtdydx\\ &= P_7+P_8.\end{aligned}$$

From this formula, we obtain $\mathbb{E}|P_7|\le Ch^4\delta t$.

For P_8, we have the following estimate:

$$\begin{aligned}P_8 &= \bar{D}_yE_xu^{2,n}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(Q^2-Q^{2,n}\Big)dtdxdy\\ &= \bar{D}_yE_xu^{2,n}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(Q^2(x,y,t)-Q^2(x_i,y_j,t_n)\Big)dtdxdy\\ &= \bar{D}_yE_xu^{2,n}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(\int_{x_i}^{x}Q_x^2(\tilde{x},y,t)d\tilde{x}\\ &\quad +\int_{y_j}^{y}Q_y^2(x_i,\tilde{y},t)d\tilde{y}+\int_{t_n}^{t}Q_s^2(x_i,y_j,s)ds\Big)dtdxdy\\ &= \bar{D}_yE_xu^{2,n}\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\Big(\int_{x_i}^{x}\int_{x_i}^{\tilde{x}}Q_{xx}^2(\tilde{\tilde{x}},y,t)d\tilde{\tilde{x}}d\tilde{x}\\ &\quad +\int_{y_j}^{y}\int_{y_j}^{\tilde{y}}Q_{yy}^2(x_i,\tilde{\tilde{y}},t)d\tilde{\tilde{y}}d\tilde{y}+\int_{t_n}^{t}Q_s^2(x_i,y_j,s)ds\Big)dtdxdy\\ &= P_{8,1}+P_{8,2}+P_{8,3}.\end{aligned}$$

Here $P_{8,3}$ can be estimated similar to $P_{6,3}$. We denote

$$P_{8,3}^B = \int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_{j+1}}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW_s^2 dtdydx \tag{C.2}$$

and then $P_{8,3}^B$ is one part of $\boldsymbol{R}_1^n$. We have

$$\mathbb{E}|P_{8,1}|,\ \mathbb{E}|P_{8,2}| \le Ch^4\delta t, \quad \mathbb{E}|P_{8,3}^A| \le Ch^2\delta t^2, \quad \mathbb{E}|P_{8,3}^B|^2 \le Ch^4\delta t^3.$$

Finally, we have the equation for the Q^2 term:

$$\begin{aligned}
V &= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(Q^2 - Q^2(x_i, y_j, t_n) \Big) dt dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{x_i}^{x} Q_x^2(\tilde{x}, y, t) d\tilde{x} \\
&+ \int_{y_j}^{y} Q_y^2(x_i, \tilde{y}, t) d\tilde{y} + \int_{t_n}^{t} Q_s^2(x_i, y_j, s) ds \Big) dt dy dx \\
&= \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \Big(\int_{x_i}^{x} \int_{x_i}^{\tilde{x}} Q_{xx}^2(\tilde{\tilde{x}}, y, t) d\tilde{\tilde{x}} d\tilde{x} \\
&+ \int_{y_j}^{y} \int_{y_j}^{\tilde{y}} Q_{yy}^2(x_i, \tilde{\tilde{y}}, t) d\tilde{\tilde{y}} d\tilde{y} + \int_{t_n}^{t} Q_s^2(x_i, y_j, s) ds \Big) dt dy dx \\
&= P_{9,1} + P_{9,2} + P_{9,3}.
\end{aligned}$$

Here $P_{9,3}$ can be estimated similar to $P_{6,3}$. We denote

$$P_{9,3}^B = \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s^2 dt dy dx \tag{C.3}$$

and then $P_{9,3}^B$ is one part of $\boldsymbol{R}_1^n$. We have

$$\mathbb{E}|P_{9,1}|,\ \mathbb{E}|P_{9,2}| \le Ch^4\delta t, \quad \mathbb{E}|P_{9,3}^A| \le Ch^2\delta t^2, \quad \mathbb{E}|P_{9,3}^B|^2 \le Ch^4\delta t^3.$$

Combining all the results above, we obtained the error estimates for remainder terms.

For $\bar{\nabla}_h \boldsymbol{R}^n$, the analysis is almost the same, only with the deference quotient replaced by integral average. Furthermore, the $\boldsymbol{R}_1^n$ terms will disappear after this difference quotient manipulation, which makes the analysis easier. The details are omitted for the lengthy statements.

Acknowledgments. The authors are grateful to Professor Weinan E, Maozheng Guo, and Yong Liu for many stimulating discussions. They also thank the referees for their careful reading and for their helpful remarks and suggestions.

REFERENCES

[1] R.B. Bird and O. Hassager, *Dynamics of Polymeric Liquids, Vol.* 1: *Fluid Mechanics*, John Wiley, New York, 1987.

[2] R.B. Bird, C.F. Curtiss, R.C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids, Vol.* 2: *Kinetic Theory*, John Wiley, New York, 1987.

[3] J. Bovin and M. Picasso, *Variance reduction methods for CONNFFESSIT-like simulations*, J. Non-Newtonian Fluid Mech., 84 (1999), pp. 191–215.

[4] M. Doi and S.F. Edwards, *The Theory of Polymer Dynamics*, Oxford University Press, Oxford, UK, 1986.

[5] W. E, T. Li, and P. Zhang, *Convergence of a stochastic method for the modeling of polymeric fluids*, Acta Math. Appl. Sin. Engl. Ser., 18 (2002), pp. 529–536.

[6] W. E, T. Li, and P. Zhang, *Well-posedness for the dumbbell model of polymeric fluids*, Comm. Math. Phys., 248 (2004), pp. 409–427.
[7] G.H. Golub and C.F. van Loan, *Matrix Computations*, 3rd ed., The Johns Hopkins University Press, Baltimore, MD, 1996.
[8] M.A. Hulsen, A.P.G. van Heel, and B.H.A.A. van den Brule, *Simulation of viscoelastic flows using Brownian configuration fields*, J. Non-Newtonian Fluid Mech., 70 (1997), pp. 79–101.
[9] B. Jourdain, T. Lelièvre, and C. Le Bris, *Numerical analysis of micro-macro simulations of polymeric fluid flows: A simple case*, Math. Models Methods Appl. Sci., 12 (2002), pp. 1205–1243.
[10] B. Jourdain, T. Lelièvre, and C. Le Bris, *Existence of solution for a micro-macro model of polymeric fluid: The FENE model*, J. Funct. Anal., 209 (2004), pp. 162–193.
[11] M. Laso and H.C. Öttinger, *Calculation of viscoelastic flow using molecular models: The CONNFFESSIT approach*, J. Non-Newtonian Fluid Mech., 47 (1993), pp. 1–20.
[12] T. Li and P. Zhang, *Mathematical and Numerical Analysis for Multi-scale Models of Complex Fluids*, http://dsec.pku.edu.cn/~tieli/tiejunli.files/publication/reviewspringer.pdf.
[13] F.-H. Lin, C. Liu, and P. Zhang, *On hydrodynamics of viscoelastic fluids*, Comm. Pure Appl. Math., 58 (2005), pp. 1437–1471.
[14] D.-G. Long, *Convergence of the random vortex method in two dimensions*, J. Amer. Math. Soc., 1 (1988), pp. 779–804.
[15] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, 5th ed., Springer-Verlag, Berlin, New York, 2000.
[16] H.C. Öttinger, *Stochastic Processes in Polymeric Liquids*, Springer-Verlag, Berlin, 1996.
[17] H.C. Öttinger, B.H.A.A. van den Brule, and M.A. Hulsen, *Brownian configuration fields and variance reduced CONNFFESSIT*, J. Non-Newtonian Fluid Mech., 70 (1997), pp. 255–261.
[18] T.A. Porsching, *Error estimates for the MAC-like approximations to the linear Navier-Stokes equations*, Numer. Math., 29 (1978), pp. 291–306.
[19] G. Strang, *Accurate partial difference methods* II. *Non-linear problems*, Numer. Math., 6 (1964), pp. 37–46.
[20] D.W. Stroock, *Probability Theory, an Analytic View*, Cambridge University Press, Cambridge, UK, 1993.