6. **Equation:** 
\[ v' = -kv^{3/2}, \quad v(0) = v_0; \quad x' = v, \quad x(0) = x_0 \]

**Solution:** 
\[
-\int \frac{dv}{2v^{3/2}} = \int \frac{1}{\sqrt{v}} \frac{dt}{2} = kt + C; \quad C = \frac{1}{\sqrt{v_0}}
\]

\[
x'(t) = v(t) = \frac{4v_0}{(2 + kt\sqrt{v_0})^2}; \quad x(t) = \frac{4v_0}{k(2 + kt\sqrt{v_0})} + C \]

\[
C = x_0 + \frac{2\sqrt{v_0}}{k}; \quad x(t) = x_0 + \frac{2\sqrt{v_0}}{k} \left(1 - \frac{2}{2 + kt\sqrt{v_0}}\right)
\]

\[ x(\infty) = x_0 + \frac{2\sqrt{v_0}}{k} \]

7. **Equation:** 
\[ v' = 10 - 0.1v, \quad x(0) = v(0) = 0 \]

(a) 
\[
\int \frac{-0.1dv}{10 - 0.1v} = \int (-0.1)dt; \quad \ln(10 - 0.1v) = -t/10 + \ln C
\]

\[ v(0) = 0 \text{ implies } C = 10; \quad \ln[(10 - 0.1v)/10] = -t/10 \]

\[ v(t) = 100(1 - e^{-t/10}); \quad v(\infty) = 100 \text{ ft/sec} \text{ (limiting velocity)} \]

(b) 
\[ x(t) = 100t - 1000(1 - e^{-t/10}) \]

\[ v = 90 \text{ ft/sec when } t = 23.0259 \text{ sec and } x = 1402.59 \text{ ft} \]

8. **Equation:** 
\[ v' = 10 - 0.001v^2, \quad x(0) = v(0) = 0 \]

(a) 
\[
\int \frac{0.01dv}{1 - 0.0001v^2} = \int \frac{dt}{10}; \quad \tanh^{-1} \frac{v}{100} = \frac{t}{10} + C
\]

\[ v(0) = 0 \text{ implies } C = 0 \text{ so } v(t) = 100\tanh(t/10) \]

\[ v(\infty) = \lim_{t \to \infty} 100\tanh(t/10) = 100 \lim_{t \to \infty} \frac{e^{t/10} - e^{-t/10}}{e^{t/10} + e^{-t/10}} = 100 \text{ ft/sec} \]

(b) 
\[ x(t) = 1000 \ln(\cosh t/10) \]

\[ v = 90 \text{ ft/sec when } t = 14.7222 \text{ sec and } x = 830.366 \text{ ft} \]

9. The solution of the initial value problem

\[ 1000 v' = 5000 - 100 v, \quad v(0) = 0 \]

is

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\[ v(t) = 50(1 - e^{-t/10}). \]

Hence, as \( t \to \infty \), we see that \( v(t) \) approaches \( v_{\text{max}} = 50 \text{ ft/sec} \approx 34 \text{ mph} \).

10. Before opening parachute:

\[

v' = -32 - 0.15v, \quad v(0) = 10000 \\
v(t) = 213.333(e^{-0.15t} - 1), \quad v(20) = -202.712 \text{ ft/sec} \\
y(t) = 11422.2 - 1422.22e^{-0.15y} - 213.333t, \quad y(20) = 7084.75 \text{ ft}
\]

After opening parachute:

\[

v' = -32 - 1.5v, \quad v(0) = -202.712, \quad y(0) = 7084.75 \\
v(t) = -21.3333 - 181.379e^{-1.5t} \\
y(t) = 6964.83 + 120.919e^{-1.5t} - 21.3333t \\
y = 0 \text{ when } t = 326.476
\]

Thus she opens her parachute after 20 sec at a height of 7085 feet, and the total time of descent is 20 + 326.476 = 346.476 sec, about 5 minutes and 46.5 seconds. Her impact speed is 21.33 ft/sec, about 15 mph.

11. If the paratrooper's terminal velocity was 100 mph = 440/3 ft/sec, then Equation (7) in the text yields \( \rho = 12/55 \). Then we find by solving Equation (9) numerically with \( y_0 = 1200 \) and \( v_0 = 0 \) that \( y = 0 \) when \( t \approx 12.5 \text{ sec} \). Thus the newspaper account is inaccurate.

12. With \( m = 640/32 = 20 \) slugs, \( W = 640 \text{ lb} \), \( B = (8)(62.5) = 500 \text{ lb} \), and \( F_R = -v \text{ lb} \) (\( F_R \) is upward when \( v < 0 \)), the differential equation is

\[
20v'(t) = -640 + 500 - v = -140 - v.
\]

Its solution with \( v(0) = 0 \) is

\[
v(t) = 140(e^{-0.05t} - 1),
\]

and integration with \( y(0) = 0 \) yields

\[
y(t) = 2800(e^{-0.05t} - 1) - 140t.
\]

Using these equations we find that \( t = 20 \ln(28/13) \approx 15.35 \text{ sec} \) when \( v = -75 \text{ ft/sec} \), and that \( y(15.35) \approx -648.31 \text{ ft} \). Thus the maximum safe depth is just under 650 ft.
\[ \frac{dv}{dt} = -g - \rho \nu^2 = -g \left(1 + \frac{\rho}{g} \nu^2 \right) = -g \left(1 + \left(\nu \sqrt{\frac{\rho}{g}} \right)^2 \right) \] (12)

Let \( u = \nu \sqrt{\frac{\rho}{g}} \). Then

\[ \frac{du}{dt} = \sqrt{\frac{\rho}{g}} \frac{dv}{dt} \quad \text{and} \quad \frac{dv}{dt} = \sqrt{\frac{g}{\rho}} \frac{du}{dt} \]

Substituting in equation (12) we get

\[ \sqrt{\frac{g}{\rho}} \frac{du}{dt} = -g(1 + u^2) \quad \text{and} \quad \frac{du}{dt} = -\sqrt{\rho g}(1 + u^2). \]

Separation of variables gives

\[ \frac{du}{1 + u^2} = -\sqrt{\rho g} \, dt \]

\[ \int \frac{du}{1 + u^2} = \int -\sqrt{\rho g} \, dt \]

\[ \tan^{-1} u = -t\sqrt{\rho g} + C \]

\[ u = \tan(C_1 - t\sqrt{\rho g}). \]

Now substituting \( u = \nu \sqrt{\frac{\rho}{g}} \) we get

\[ \nu \sqrt{\frac{\rho}{g}} = \tan(C_1 - t\sqrt{\rho g}). \]

\[ \nu = \sqrt{\frac{g}{\rho}} \tan(C_1 - t\sqrt{\rho g}). \]

The initial condition \( v(0) = v_0 \) gives \( C_1 = \tan^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right) \).

\[ v(t) = \sqrt{\frac{g}{\rho}} \tan(C_1 - t\sqrt{\rho g}) \quad \text{with} \quad C_1 = \tan^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right). \] (13)
\[ v(t) = \sqrt{\frac{g}{\rho}} \tan \left( C_1 - t \sqrt{\rho g} \right) \quad \text{with} \quad C_1 = \tan^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right) \quad (13) \]

Integrating both sides yields

\[ \int v(t) = \int \sqrt{\frac{g}{\rho}} \tan \left( C_1 - t \sqrt{\rho g} \right) \]

\[ y(t) = \sqrt{\frac{g}{\rho}} \left( \frac{-1}{\sqrt{\rho g}} \right) \left( - \ln |\cos(C_1 - t \sqrt{\rho g})| \right) + C' \]

and simplifying this results

\[ y(t) = \frac{1}{\rho} \ln |\cos(C_1 - t \sqrt{\rho g})| + C' \]

Using the initial condition \( y(0) = y_0 \) we get

\[ y(0) = \frac{1}{\rho} \ln |\cos C_1| + C_2 \]

\[ y_0 = \frac{1}{\rho} \ln |\cos C_1| + C_2 \]

\[ C' = y_0 - \frac{1}{\rho} \ln |\cos C_1| \]

Therefore

\[ \dot{y}(t) = \frac{1}{\rho} \ln |\cos(C_1 - t \sqrt{\rho g})| + (y_0 - \frac{1}{\rho} \ln |\cos C_1|) \]

\[ y(t) = y_0 + \frac{1}{\rho} \left( \ln |\cos(C_1 - t \sqrt{\rho g})| - \ln |\cos C_1| \right) \]

\[ y(t) = y_0 + \frac{1}{\rho} \ln \left| \frac{\cos(C_1 - t \sqrt{\rho g})}{\cos C_1} \right| \quad (14) \]
\[ \frac{dv}{dt} = -g + \rho v^2 = -g \left( 1 - \frac{\rho}{g} v^2 \right) = -g \left( 1 - \left( v \sqrt{\frac{\rho}{g}} \right)^2 \right) \]  \hspace{1cm} (15)

Let \( u = v \sqrt{\frac{\rho}{g}} \). Then

\[ \frac{du}{dt} = \sqrt{\frac{\rho}{g}} \frac{dv}{dt} \quad \text{and} \quad \frac{dv}{dt} = \sqrt{\frac{g}{\rho}} \frac{du}{dt} \]

Substituting in equation (15) we get

\[ \sqrt{\frac{g}{\rho}} \frac{du}{dt} = -g (1 - u^2) \quad \text{and} \quad \frac{du}{dt} = -\sqrt{\rho g} (1 - u^2). \]

Separation of variables gives

\[ \frac{du}{1 - u^2} = -\sqrt{\rho g} \, dt \]

\[ \int \frac{du}{1 - u^2} = \int -\sqrt{\rho g} \, dt \]

\[ \tanh^{-1} u = -t \sqrt{\rho g} + C \]

\[ u = \tanh(C_2 - t \sqrt{\rho g}). \]

Now substituting \( u = v \sqrt{\frac{\rho}{g}} \) we get

\[ v \sqrt{\frac{\rho}{g}} = \tanh(C_2 - t \sqrt{\rho g}); \]

\[ v = \sqrt{\frac{g}{\rho}} \tanh(C_2 - t \sqrt{\rho g}). \]

The initial condition \( v(0) = v_0 \) gives \( C_2 = \tanh^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right) \).

\[ v(t) = \sqrt{\frac{g}{\rho}} \tanh(C_2 - t \sqrt{\rho g}) \quad \text{with} \quad C_2 = \tanh^{-1} \left( v_0 \sqrt{\frac{\rho}{g}} \right). \]
\[ v(t) = \sqrt{\frac{g}{\rho}} \tanh(C_2 - t\sqrt{\rho g}) \quad \text{with} \quad C_2 = \tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right) \quad (16) \]

Integrating both sides yields

\[
\int v(t) = \int \sqrt{\frac{g}{\rho}} \tanh(C_2 - t\sqrt{\rho g})
\]

\[ y(t) = \sqrt{\frac{g}{\rho}} \left( -\frac{1}{\sqrt{\rho g}} \right) (\ln |\cosh(C_2 - t\sqrt{\rho g})|) + C \]

and simplifying this results

\[ y(t) = -\frac{1}{\rho} \ln |\cosh(C_2 - t\sqrt{\rho g})| + C. \]

Using the initial condition \( y(0) = y_0 \) we get

\[ y(0) = -\frac{1}{\rho} \ln |\cosh C_2| + C \]

\[ y_0 = -\frac{1}{\rho} \ln |\cosh C_2| + C \]

\[ C = y_0 + \frac{1}{\rho} \ln |\cosh C_2|. \]

Therefore

\[ y(t) = -\frac{1}{\rho} \ln |\cosh(C_2 - t\sqrt{\rho g})| + (y_0 + \frac{1}{\rho} \ln |\cosh C_2|) \]

\[ y(t) = y_0 - \frac{1}{\rho} (\ln |\cosh(C_2 - t\sqrt{\rho g})| - \ln |\cosh C_2|) \]

\[ y(t) = y_0 - \frac{1}{\rho} \ln \left| \frac{\cosh(C_2 - t\sqrt{\rho g})}{\cosh C_2} \right|. \quad (17) \]
Given the hints and integrals provided in the text, Problems 13–16 are fairly straightforward (and fairly tedious) integration problems.

17. To solve the initial value problem \( v' = -9.8 - 0.0011v^2, \quad v(0) = 49 \), we write

\[
\int \frac{dv}{9.8 + 0.0011v^2} = -\int dt; \quad \int \frac{0.010595 \, dv}{1 + (0.010595v)^2} = -\int 0.103827 \, dt
\]

\[\tan^{-1}(0.010595v) = -0.103827t + C; \quad v(0) = 49 \implies C = 0.478854\]

\[v(t) = 94.3841 \tan(0.478854 - 0.103827t)\]

Integration with \( y(0) = 0 \) gives

\[y(t) = 108.468 + 909.052 \ln(\cos(0.478854 - 0.103827t))\].

We solve \( v(0) = 0 \) for \( t = 4.612 \), and then calculate \( y(4.612) = 108.468 \).

18. We solve the initial value problem \( v' = -9.8 + 0.0011v^2, \quad v(0) = 0 \) much as in Problem 17, except using hyperbolic rather than ordinary trigonometric functions. We first get

\[v(t) = -94.3841 \tanh(0.103827t),\]

and then integration with \( y(0) = 108.47 \) gives

\[y(t) = 108.47 - 909.052 \ln(\cosh(0.103827t))\].

We solve \( y(0) = 0 \) for \( t = \cosh^{-1}(\exp(108.47/909.052))/0.103827 \approx 4.7992 \), and then calculate \( v(4.7992) = -43.489 \).

19. Equation: \( v' = 4 - (1/400)v^2, \quad v(0) = 0 \)

Solution: \( \int \frac{dv}{4 - (1/400)v^2} = \int dt; \quad \int \frac{(1/40)}{1 - (v/40)^2} \, dv = \int \frac{1}{10} \, dt \)

\[\tanh^{-1}(v/40) = t/10 + C; \quad C = 0; \quad v(t) = 40 \tanh(t/10)\]

Answer: \( v(10) \approx 30.46 \, \text{ft/sec}, \quad v(\infty) = 40 \, \text{ft/sec}\)

20. Equation: \( v' = -32 - (1/800)v^2, \quad v(0) = 160, \quad y(0) = 0 \)

Solution: \( \int \frac{dv}{32 + (1/800)v^2} = -\int dt; \quad \int \frac{(1/160)v}{1 + (v/160)^2} \, dv = -\int \frac{1}{5} \, dt; \)

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\[
\tan^{-1}(v/160) = -t/5 + C; \quad v(0) = 160 \text{ implies } C = \pi/4
\]

\[
v(t) = 160 \tan\left(\frac{\pi}{4} - \frac{t}{5}\right)
\]

\[
y(t) = 800 \ln\left(\cos\left(\frac{\pi}{4} - \frac{t}{5}\right)\right) + 400 \ln 2
\]

We solve \(v(t) = 0\) for \(t = 3.92699\) and then calculate \(y(3.92699) = 277.26\) ft.

21. Equation: \(v' = -g - \rho v^2; \quad v(0) = v_0, \quad y(0) = 0\)

Solution: \(\int \frac{dv}{g + \rho v^2} = - \int dt; \quad \int \frac{\sqrt{\rho / g} \, dv}{1 + \left(\sqrt{\rho / g} v\right)^2} = - \int \sqrt{g \rho} \, dt; \quad \tan^{-1}\left(\sqrt{\rho / g} v\right) = -\sqrt{g \rho} \, t + C; \quad v(0) = v_0 \text{ implies } C = \tan^{-1}\left(\sqrt{\rho / g} v_0\right)\)

\[
v(t) = -\sqrt{\frac{g}{\rho}} \tan\left(t \sqrt{g \rho} - \tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right)\right)
\]

We solve \(v(t) = 0\) for \(t = \frac{1}{\sqrt{g \rho}} \tan^{-1}\left(v_0 \sqrt{\frac{\rho}{g}}\right)\) and substitute in Eq. (17) for \(y(t)\):

\[
y_{\text{max}} = \frac{1}{\rho} \ln \left|\frac{\cos\left(\tan^{-1}v_0 \sqrt{\rho / g} - \tan^{-1}v_0 \sqrt{\rho / g}\right)}{\cos\left(\tan^{-1}v_0 \sqrt{\rho / g}\right)}\right|
\]

\[
= \frac{1}{\rho} \ln\left(\sec\left(\tan^{-1}v_0 \sqrt{\rho / g}\right)\right) = \frac{1}{\rho} \ln\left(1 + \frac{\rho v_0^2}{g}\right)
\]

\[
y_{\text{max}} = \frac{1}{2 \rho} \ln\left(1 + \frac{\rho v_0^2}{g}\right)
\]

22. By an integration similar to the one in Problem 19, the solution of the initial value problem \(v' = -32 + 0.075v^2; \quad v(0) = 0\) is

\[
v(t) = -20.666 \tanh(1.54919t),
\]

so the terminal speed is 20.666 ft/sec. Then a further integration with \(y(0) = 0\) gives

\[
y(t) = 10000 - 13.333 \ln(\cosh(1.54919t)).
\]

We solve \(y(0) = 0\) for \(t = 484.57\). Thus the descent takes about 8 min 5 sec.
23. Before opening parachute:

\[
\begin{align*}
\dot{v'} &= -32 + 0.00075v'^2, \quad v(0) = 0, \quad y(0) = 10000 \\
v(t) &= -206.559 \tanh(0.154919t) \quad v(30) = -206.521 \text{ ft/sec} \\
y(t) &= 10000 - 1333.33 \ln(\cosh(0.154919t)), \quad y(30) = 4727.30 \text{ ft}
\end{align*}
\]

After opening parachute:

\[
\begin{align*}
\dot{v'} &= -32 + 0.075v'^2, \quad v(0) = -206.521, \quad y(0) = 4727.30 \\
v(t) &= -20.6559 \tanh(1.54919t + 0.00519595) \\
y(t) &= 4727.30 - 13.33333 \ln(\cosh(1.54919t + 0.00519595)) \\
y = 0 \text{ when } t = 229.304
\end{align*}
\]

Thus she opens her parachute after 30 sec at a height of 4727 feet, and the total time of descent is \(30 + 229.304 = 259.304\) sec, about 4 minutes and 19.3 seconds.

24. Let \(M\) denote the mass of the Earth. Then

(a) \(\sqrt{2GM/R} = c\) implies \(R = 0.884 \times 10^{-3}\) meters, about 0.88 cm;

(b) \(\sqrt{2G(329320M)/R} = c\) implies \(R = 2.91 \times 10^3\) meters, about 2.91 kilometers.

25. (a) The rocket's apex occurs when \(v = 0\). We get the desired formula when we set \(v = 0\) in Eq. (23),

\[
v^2 = v_0^2 + 2GM \left( \frac{1}{r} - \frac{1}{R} \right),
\]

and solve for \(r\).

(b) We substitute \(v = 0, \quad r = R + 10^5 \) (100 km = 10^5 m) and the mks values
\(G = 6.6726 \times 10^{-11}, \quad M = 5.975 \times 10^{24}, \quad R = 6.378 \times 10^6\) in Eq. (23) and solve for \(v_0 = 1389.21\) m/s \(\approx 1.389\) km/s.

(c) When we substitute \(v_0 = (9/10)\sqrt{2GM/R}\) in the formula derived in part (a), we find that \(r_{max} = 100R/19\).

26. By an elementary computation (as in Section 1.2) we find that an initial velocity of \(v_0 = 16\) ft/sec is required to jump vertically 4 feet high on earth. We must determine whether this initial velocity is adequate for escape from the asteroid. Let \(r\) denote the ratio of the radius of the asteroid to the radius \(R = 3960\) miles of the earth, so that

\[
r = \frac{1.5}{3960} = \frac{1}{2640}.
\]