1(20 points): In each of the following five problems, the roots of the characteristic equation of a certain homogenous differential equation are given. Write the general solution of the homogeneous differential equation. For each of the following problems, \( x \) is the independent variable, and \( y(x) \) is the unknown function. You can use \( c_1, c_2, \cdots \) to denote the arbitrary constants in the expression of the general solution.

1. Roots: \(-2, -1/3, -1/3, 0, \sqrt{2}\)
   General solution: \(c_1 e^{-2x} + c_2 e^{-\frac{x}{3}} + c_3 x e^{-\frac{x}{3}} + c_4 + c_5 e^{\sqrt{2}x}\)

2. Roots: \(3i, -3i, 3i, -3i, 3i, -3i\)
   General solution: \(c_1 \cos 3x + c_2 \sin 3x + c_3 x \cos 3x + c_4 x \sin 3x + c_5 x^2 \cos 3x + c_6 x^2 \sin 3x\)

3. Roots: \(2 + 7i, 2 + 7i, 2 - 7i, 2 - 7i\)
   General solution: \(c_1 e^{2x} \cos 7x + c_2 e^{2x} \sin 7x + c_3 x e^{2x} \cos 7x + c_4 x e^{2x} \sin 7x\)

4. Roots: \(0, 0, 3, -2 + 3i, -2 - 3i\)
   General solution: \(c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 e^{-2x} \cos 3x + c_6 e^{-2x} \sin 3x\)

5. Roots: \(-1, 0, 3, 3 + \sqrt{7}i, 3 - \sqrt{7}i\)
   General solution: \(c_1 e^{-x} + c_2 + c_3 x + c_4 e^{3x} + c_5 e^{3x} \cos 7x + c_6 e^{3x} \sin 7x\)

2. (20 points) In each of the following five problems, set up the appropriate form of the a particular solution \(y_p\), but DO NOT determine the values of the coefficients. You can use \(A, B, C, \cdots\) to denote the coefficients. (Hint: operator \(D\) is explained at the end of the problem.)
1. \((\mathbb{D} + 2)(\mathbb{D} + 1)y = e^{2x} + 3e^{3x} + x^2e^x\)
   
   \[y_p = Ae^{2x} + Be^{3x} + (C + Dx + Ex^2)e^x\]

2. \((\mathbb{D} + 2)(\mathbb{D} + 1)y = \sin 2x + \sin 3x\)
   
   \[y_p = A\cos 2x + B\sin 2x + C\cos 3x + D\sin 3x\]

3. \(\mathbb{D}^2(\mathbb{D} + 2)(\mathbb{D} + 1)y = 3x^2 + \cos(3x)\)
   
   \[y_p = x^2(A + Bx + Cx^2) + D\cos 3x + E\sin 3x\]

4. \((\mathbb{D}^2 + 4)y = x\sin 2x + e^x\)
   
   \[y_p = x[(A + Bx)\cos 2x + (C + Dx)\sin 2x] + Ee^x\]

5. \((\mathbb{D} + 2)(\mathbb{D} + 1)y = e^{2x}\cos(2x)\)
   
   \[y_p = Ae^{2x}\cos 2x + Be^{2x}\sin 2x\]

(Hint: In this problem, we suppose that \(y(x)\) is the unknown function and \(x\) is the independent variable. \(\mathbb{D}\) is an operator denoting the operation of differentiation with respect to \(x\), so that

\[\mathbb{D} y = \frac{dy}{dx} = y'\]

We know it is easy to get the characteristic equation for differential equations written by using the operator \(\mathbb{D}\), e.g. the characteristic equation of

\[(\mathbb{D} - 2)^2(\mathbb{D} + 3)y = 0\]

is

\[(r - 2)^2(r + 3) = 0\).
3(15 points): Solve the following initial value problem \((x)\) is the independent variable):
\[ y'' + 4y = 8x^2; \quad y(0) = 1, \quad y'(0) = 0. \]

Solution: The characteristic equation is
\[ r^2 + 4 = 0 \]
and its roots are \( \pm 2i \). So the complementary function is
\[ y_c = c_1 \cos 2x + c_2 \sin 2x \]
The first guess of a particular solution is of the form
\[ y_p = A + Bx + Cx^2 \]
according to the right hand of the nonhomogeneous equation. Compare \( y_c \) and \( y_p \) and we find there is no duplication. So the appropriate form of a particular solution is
\[ y_p = A + Bx + Cx^2 \]
Substitution \( y_p \) and its derivatives into the nonhomogeneous equation yields
\[ 2C + 4(A + Bx + Cx^2) = 8x^2 \]
Collecting coefficients and equating the coefficients of like terms yield
\[ 2C + 4A = 0; \quad 4B = 0; \quad 4C = 8 \]
So \( A = -1, \ B = 0, \ C = 2 \). So the general solution of the nonhomogeneous equation is
\[ y = c_1 \cos 2x + c_2 \sin 2x + (-1 + 2x^2) \]
The initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \) yield
\[ 1 = c_1 \cos(2 \cdot 0) + c_2 \sin(2 \cdot 0) + (-1 + 2 \cdot 0^2) \]
\[ 0 = -2c_1 \sin(2 \cdot 0) + 2c_2 \cos(2 \cdot 0) + 4 \cdot 0 \]
So \( c_1 = 2, \ c_2 = 0 \). Plugging \( c_1 \) and \( c_2 \) into the general solution yields the solution of the initial value problem
\[ y = -1 + 2x^2 + 2 \cos 2x \]
4. (15 points) Use the method of variation of parameters to find a particular solution of the given differential equation ($x$ is the independent variable)

\[ y'' - 2y' + 2y = e^x \sec x. \]

Solution: The characteristic equation $r^2 - 2r + 2 = 0$ has roots $1 \pm i$. So the complementary function is

\[ y_c = c_1 e^x \cos x + c_2 e^x \sin x \]
\[ y_1 = e^x \cos x, \quad y_2 = e^x \sin x \]

The Wronskian is

\[ W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = e^{2x} \]

\[ u'_1(x) = - \frac{y_2(x)e^x \sec x}{W(x)} = - \frac{e^x \sin x \sec x}{e^x} = - \tan x \]

\[ u'_2(x) = \frac{y_1(x)e^x \sec x}{W(x)} = - \frac{e^x \cos x \sec x}{e^x} = 1 \]

\[ u_1(x) = \int - \tan x \, dx = \ln |\cos x| \]

\[ u_2(x) = \int 1 \, dx = x \]

So we get a particular solution

\[ y = u_1(x)y_1(x) + u_2(x)y_2(x) = \ln |\cos x| e^x \cos x + xe^x \sin x \]
5. (10 points): Transform the given equation ($t$ is the independent variable) into an equivalent system of first-order differential equations.

$$x^{(4)} = 3x^{(3)} + 5x'' + 3x + \cos 4t.$$ 

Solution: Let $x_1 = x, x_2 = x', x_3 = x'', x_4 = x^{(3)}$. 

Equivalent system:

\begin{align*}
x_1' &= x_2 \\
x_2' &= x_3 \\
x_3' &= x_4 \\
x_4' &= 3x_4 + 5x_3 + 3x_1 + \cos 4t
\end{align*}
6. (15 points) Find general solutions of the given linear system \((t\) is the independent variable) by using the method of elimination.

\[
x' = 2x + y, \quad y' = x + 2y - e^{2t}
\]

Solution:

\[
x' = 2x + y \quad (1)
\]

\[
y' = x + 2y - e^{2t} \quad (2)
\]

From Eq. (1), we have

\[
y = x' - 2x \quad (3)
\]

and

\[
y' = x'' - 2x' \quad (4)
\]

Plugging Eq. (3) and Eq. (4) into Eq. (2) yields

\[
x'' - 2x' = x + 2(x' - 2x) - e^{2t}
\]

which is simplified into

\[
x'' - 4x' + 3x = -e^{2t} \quad (5)
\]

The characteristic equation of Eq. (5) \(r^2 - 4r + 4 = 0\) has roots 1 and 3. So the complementary function of Eq. (5) is

\[
x_c = c_1 e^t + c_2 e^{3t} \quad (6)
\]

First guess of the form of a particular solution is

\[
x_{\text{trial}} = Ae^{2t} \quad (7)
\]

Comparing \(x_c\) and \(x_{\text{trial}}\), we find there is no duplication and get the appropriate form of a particular solution

\[
x_p = Ae^{2t} \quad (8)
\]

Plugging Eq. (8) into Eq. (5) yields

\[
4Ae^{2t} - 4 \cdot (2Ae^{2t}) + 3Ae^{2t} = -e^{2t}
\]

which gives \(A = 1\). We plug \(A = 1\) into Eq. (8) and get a particular solution

\[
x_p = e^{2t}
\]

So the general solution for \(x\) is

\[
x = c_1 e^t + c_2 e^{3t} + e^{2t} \quad (10)
\]

And we can get from Eq. (10)

\[
x' = c_1 e^t + 3c_2 t e^{3t} + 2e^{2t} \quad (11)
\]

Plugging Eqs. \((refeq10)\) and (11) into Eq. (3) yields the general solution for \(y\)

\[
y = -c_1 e^t + 2c_2 e^{3t} \quad (12)
\]

So the general solution of the system is (10) and (12).
Use the Wronskian to prove the functions

\[ f(x) = e^x, \quad g(x) = e^{2x}, \quad h(x) = e^{3x} \]

are linearly independent on the real line.

Solution: Wronskian of the three functions is

\[
W(f(x), g(x), h(x)) = \begin{vmatrix}
    e^x & e^{2x} & e^{3x} \\
    e^x & 2e^{2x} & 3e^{3x} \\
    e^x & 4e^{2x} & 9e^{3x}
\end{vmatrix}
\]

\[
= e^{(x+2x+3x)} \begin{vmatrix}
    1 & 1 & 1 \\
    1 & 2 & 3 \\
    1 & 4 & 9
\end{vmatrix}
\]

\[
= e^{6x} \begin{vmatrix}
    1 & 1 & 1 \\
    0 & 1 & 2 \\
    0 & 3 & 8
\end{vmatrix}
\]

\[
= e^{6x} \begin{vmatrix}
    1 & 1 & 1 \\
    0 & 1 & 2 \\
    0 & 0 & 2
\end{vmatrix}
= 2e^{6x} \neq 0
\]

So the three functions are linearly independent on the real line.