Boltzmann equation

Boltzmann equation reads
\[ \frac{df}{d\tau} + \nabla \cdot (Q(f, f)) = 0. \]  

- \( f \rightarrow \) phase density distribution function
- \( Q(f, f) \rightarrow \) collision term
- \( \tau \rightarrow \) time
- \( \nabla \rightarrow \) gradient operator

\( f \in \mathbb{R}^D \)

Difficulties of direct numerical simulation of Boltzmann equation:

- high-order variable: \( 1 + D + D = 2D + 1 \)
- infinite speeds \( v \in \mathbb{R}^D \)

\[ \text{Boltzmann equation} \implies \text{Hydrodynamic equations} \]

Levermore’s maximum entropy:

\[ \text{Maximum entropy:} \]

\[ \max_T \{ f(T) - f \}, \quad \text{subject to} \quad \langle \phi \rangle = m_i, \]

\[ m_i \rightarrow \text{moments we concern about.} \]

\[ \alpha \rightarrow \text{Lagrange multiplier.} \]

\[ \text{Moment system: Can’t be written into analysis form except 5 and 10 moment systems.} \]

Grad’s moment method and Levermore’s maximum entropy (ME)

\[ \text{Grad’s moment method:} \]

\[ \text{Idea: Consider distribution} f \text{ is far from equilibrium} f \sim f_{eq}. \]

\[ \text{Ansatz: Gradient expansion:} \]

\[ f \approx f_{Gr} = f_{eq} \sum_i \alpha_i \phi_i(q). \]

\[ \phi_i \rightarrow \text{multi-variable polynomials.} \]

\[ \text{Moment system: Substituting} f_{Gr} \text{ into Boltzmann equation } \]

\[ \text{and matching coefficients of} \ phi_i \text{to obtain equations of} \ a_i \text{ and} \ \alpha_i. \]

Levermore’s maximum entropy:

\[ \text{Idea: Maximum entropy:} \]

\[ \max_T \{ f(T) - f \}, \quad \text{subject to} \quad \langle \phi \rangle = m_i, \]

\[ m_i \rightarrow \text{moments we concern about.} \]

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\[ \text{Moment system:} \]

\[ \sim \text{Can’t be written into analysis form except 5 and 10 moment systems.} \]

Comparison of Grad’s and Levermore’s moment system:

<table>
<thead>
<tr>
<th>Properties</th>
<th>Grad’s</th>
<th>Levermore’s</th>
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<tbody>
<tr>
<td>Physical properties</td>
<td>( \checkmark )</td>
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</tr>
<tr>
<td>Model properties</td>
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<tr>
<td>Model complexity</td>
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<td>( \checkmark )</td>
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<tr>
<td>Numerical properties</td>
<td>( \checkmark )</td>
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</table>

**Generalized Hermite moment system**

- 1. Generalized Hermite moment equation (GHME) is Galilean transformation invariant.
- 2. Each characteristic wave is in simple wave, i.e. either genuinely nonlinear or linearly degenerate.
- 3. Relationship of wave type and values on both sides, similar as that of Euler equations and 10 moment system:

\[ \text{Wave type} \]

\[ \text{Equation} = \text{Gradient expansion} \]

\[ \text{Moment wave} \]

\[ \text{Contact wave} \]

\[ \text{Similar as that of Euler equations.} \]

Comparison with Grad’s moment system: numerical results

In the numerical examples, BGS(k) collision term is used, with relaxation time \( \tau = Kn/t \).

**Shock tube**

Initial value: \( \begin{cases} (0, t) & \tau = 0 \tau < 0 \, \text{or} \, \tau > 0 \end{cases} \)

\( Kn = 0.5 \)

\( Kn = 5 \)

**Conclusion & future work**

- We proposed a new series moment system based on Generalized Hermite expansion, which is similar as Grad’s expansion.
- The weight function of Generalized Hermite moment system contains some non-equilibrium information of distribution function.
- Numerical results show even for 1D flow, GHME behaves better than Grad’s moment system.

**Future work**

- Numerical simulation for 2D and 3D flow.
- Boundary condition and analysis on boundary layer.

**References**


