ADAPTIVE FINITE ELEMENT APPROXIMATION FOR DISTRIBUTED OPTIMAL CONTROL GOVERNED BY PARABOLIC EQUATIONS

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Abstract. In this paper, we study adaptive finite element algorithms for distributed convex optimal control problem governed by a linear parabolic equation with an obstacle constraint. We first examine the backward Euler scheme, and then derive sharp a posteriori error estimates for the scheme. Due to the global nature of the control problem, we also examine the finite element approximation with an isotropic treatment of the variables \( x \) and \( t \). We derive sharp a posteriori error estimates for both the state and the control approximation. These estimates are then used as indicators in adaptive finite element approximation schemes for the control problem. Using a preconditioned project gradient algorithm, we conduct several numerical experiments to compare the two schemes. It was found that the second scheme normally results in much better mesh-refinement. It is normally more efficient when the optimal control is not smooth.

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1. Introduction

Finite element approximation of optimal control problems plays an important role in numerical methods of these problems. There have been extensive studies in this aspect, see, for example, [2, 3, 11, 12, 13, 14, 15, 20, 22, 24], [33, 34, 35, 36, 38, 39, 40], and the references cited therein. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [9, 15, 35, 38].

In order to obtain a numerical solution of acceptable accuracy for the optimal control problem, the finite element meshes have to be refined according to a mesh refinement scheme. Adaptive finite element approximation uses a posteriori error indicator to guide the mesh refinement procedure. Only the area where the error indicator is larger will be refined so that a higher density of nodes is distributed over the area where the solution is difficult to approximate. In this sense efficiency and reliability of adaptive finite element approximation rely very much on the error indicator used.

Recently adaptive mesh refinement has been found quite useful in computing optimal control problems governed by elliptic systems, see [4], [5], [29], [30], and [32], for example. Usually the optimal control has only limited regularity (say, at most in \( H^1(\Omega) \) in general). Thus suitable adaptive meshes

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can quite efficiently reduce the control approximation error. The details can be found in [4], [5], [17] and [25]. Particularly it seems to be important to use multi-set adaptive meshes in applying adaptive finite element method to computing optimal control, see [17]. However it is much more complicated to implement adaptive computational schemes for evolutional control problems.

In computing numerical solutions for a parabolic equation, time matching schemes like the backward Euler scheme are mostly used. The main advantage of using the backward Euler scheme in solving parabolic equations is that one can then solve them step by step in time. However that also means it is impossible to adjust mesh locally in \( x-t \) space. Consequently an adaptive backward Euler scheme may not be able to properly handle the singularities that have complicated geometric distribution in \( x-t \) space. Due to the global property of the control problem in time it may be less important to use time matching discretization. Thus after examining the backward Euler scheme for the control problem, we propose a new scheme which approximates the state and the control by the finite element method with an isotropic treatment of the variables \( x \) and \( t \). In our numerical tests, we found that for a given error tolerance, the computational work on solving the full linear systems generated from the new scheme is often much smaller than that on solving the backward Euler systems, unless the optimal control is smooth.

For both the schemes, we derive a posteriori error estimates for the state and the control approximation, for the case of an obstacle constraint, which is most frequently met in applications. These estimators are much sharper than those obtained in [31] for general constraints, see [25]. They are then used as the error indicators in adaptive finite element approximation schemes for the control problem, and numerical results are presented.

The plan of the paper is as follows: In Section 2 we first examine the backward Euler scheme in time and standard finite element approximation in space for the control problem, and then propose a new approximation scheme for the optimal control problem. In Section 3 a posteriori error estimators are derived for the control problem using the backward Euler scheme. In Section 4 a posteriori error estimates are obtained for the problem using the new approximation scheme. In Section 5, we discuss a preconditioned projection algorithm to solve the control problem. Section 6 is devoted to numerical experiments.

Let \( \Omega \) and \( \Omega_v \) bounded open sets in \( \mathbb{R}^n \) \((n \leq 3)\) with Lipschitz boundaries \( \partial \Omega \) and \( \partial \Omega_v \). In this paper, we adopt the standard notation \( W^{m,q}(\Omega) \) for Sobolev spaces on \( \Omega \) with norm \( \| \cdot \|_{m,q,\Omega} \) and seminorm \( | \cdot |_{m,q,\Omega} \). We denote \( W^{m,2}(\Omega) \) by \( H^m(\Omega) \) and set \( H^1_0(\Omega) \equiv \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \} \). We denote by \( L^s(0,T; W^{m,q}(\Omega)) \) the Banach space of all \( L^s \) integrable functions from \((0,T)\) into \( W^{m,q}(\Omega)\) with norm \( \| v \|_{L^s(0,T; W^{m,q}(\Omega))} = \left( \int_0^T \| v(t) \|_{W^{m,q}(\Omega)}^s \, dt \right)^{1/s} \) for \( s \in [1,\infty) \) and the standard modification for \( s = \infty \). Similarly, one define the spaces \( H^1(0,T; W^{m,q}(\Omega)) \) and \( C^0(0,T; W^{m,q}(\Omega)) \). The details can be found in [27]. In addition \( c \) or \( C \) denotes a general positive constant independent of the mesh size parameter \( h \).

2. Approximation Scheme of Optimal Control Governed by Parabolic Equations

In this section, we study two numerical schemes to approximate distributed convex optimal control problems governed by a parabolic equation. We shall take the state space \( W = L^2(0,T; Y) \) with \( Y = H^1_0(\Omega) \), the control space \( X = L^2(0,T; U) \) with \( U = L^2(\Omega_v) \) and \( H = L^2(\Omega) \) to fix the idea. Let \( B \) be a linear continuous operator from \( X \) to \( L^2(0,T; Y^*) \). Let \( K \) be a closed convex set in \( U \). Let \( g \) be a convex functional which is continuously differentiable on the observation space \( H = L^2(\Omega) \),
and $h()$ be a strictly convex continuously differentiable functional on $U$. We further assume that $h(u) \to +\infty$ as $\|u\|_U \to \infty$ and that the functional $g()$ is bounded below. We are interested in the following optimal control problem:

$$
\min_{u \in X, y \in K} \left\{ \int_0^T (g(y) + h(u)) \, dt \right\},
$$

with boundary conditions

$$
\begin{align*}
\frac{\partial y}{\partial t} - \text{div}(A\nabla y) &= f + Bu, & x \in \Omega, & t \in (0,T], \\
y|_{\partial \Omega} &= 0, & t \in [0,T], \\
y(x,0) &= y_0(x), & x \in \Omega,
\end{align*}
$$

where $f \in L^2(0,T; L^2(\Omega))$, $y_0 \in H^1_0(\Omega)$, and

$$
A(x) = (a_{ij}(x))_{n \times n} \in (C^\infty(\bar{\Omega}))^{n \times n}
$$

such that there is a constant $c > 0$ satisfying

$$
(A\xi) \cdot \xi \geq c|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.
$$

Let

$$
a(v, w) = \int_\Omega (A\nabla v) \cdot \nabla w, \quad \forall v, w \in H^1(\Omega),
$$

$$
(f_1, f_2) = \int_\Omega f_1 f_2, \quad \forall f_1, f_2 \in L^2(\Omega),
$$

$$
(v, w)_v = \int_{\Omega_v} vw, \quad \forall v, w \in L^2(\Omega_v).
$$

It follows from the assumptions on $A$ that there are constants $c$ and $C > 0$ such that

$$
a(v, v) \geq c\|v\|_{H^1(\Omega)}^2, \quad |a(v, w)| \leq C|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}, \quad \forall v, w \in Y.
$$

Then a weak formula of the convex optimal control problem reads:

$$
(2.1) \quad \min_{u \in K} \left\{ \int_0^T (g(y) + h(u)) \, dt \right\},
$$

where $y \in W, u \in X, u(t) \in K$ subject to

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial y}{\partial t} - \text{div}(A\nabla y) &= f + Bu, \quad \forall w \in Y, \quad t \in (0,T], \\
y(0) &= y_0.
\end{array} \right.
\end{align*}
$$

It is well known (see, e.g., [26]) that the control problem (2.1) has a unique solution $(y, u)$, and that a pair $(y, u)$ is the solution of (2.1) if and only if there is a co-state $p \in W$ such that the triplet $(y, p, u)$ satisfies the following optimality conditions:

$$
(2.2) \quad \begin{cases}
\left( \begin{array}{l}
\frac{\partial y}{\partial t} + a(y, w) = f + Bu, & \forall w \in Y, \\
-p & = a(q, p) = (g(q), q), & \forall q \in Y,
\end{array} \right.
\end{cases} \\
\int_0^T (h'(u) + B^*p, v - u_v) \, dt \geq 0, \quad \forall (v, t) \in K, v \in X = L^2(0,T; U),
$$

where $B^*$ is the adjoint operator of $B$, $g'$ and $h'$ are the derivatives of $g$ and $h$, which have been viewed as functions in $H = L^2(\Omega)$ and $X = L^2(0,T; L^2(\Omega_v))$, respectively, and $(\cdot, \cdot)_v$ is the inner product of $U$, which will be simply written as $(\cdot, \cdot)$ in the rest of the paper whenever no confusion should be caused. For the details of these results, we refer to [26, 35].
Let us consider the finite element approximation of the control problem (2.1). We first consider the standard backward Euler scheme. Let $\Omega^h$ be a polygonal approximation to $\Omega$ with boundary $\partial \Omega^h$. Let $T^h$ be a partitioning of $\Omega^h$ into disjoint regular $n$-simplices $\tau$, so that $\Omega^h = \bigcup_{\tau \in T^h} \tau$. Each element has at most one face on $\partial \Omega^h$, and $\tau$ and $\tau'$ have either only one common vertex or a whole edge or face if $\tau$ and $\tau'$ are in $T^h$. We further require that $P_i \in \partial \Omega^h \Rightarrow P_i \in \partial \Omega$ where $\{P_i\}_{i=1,\ldots,J}$ is the vertex set associated with the triangulation $T^h$. Just for simplicity, we assume that $\Omega$ is a convex polygon so that $\Omega = \Omega^h$.

Associated with $T^h$ is a finite dimensional subspace $S^h$ of $C(\Omega^h)$, such that $\chi_{\tau}$ are polynomials of m-order ($m \geq 1$) for all $\chi \in S^h$ and $\tau \in T^h$. Let $V^h = \{ v_h \in S^h : v_h|_{\partial \Omega} = 0 \}$, $W^h = L^2(0,T; V^h)$. It is easy to see that $V^h \subset Y$, $W^h \subset W$.

Let $T^h_U$ be a partitioning of $\Omega^h_U$ into disjoint regular $n$-simplices $\tau_U$, so that $\Omega^h_U = \bigcup_{\tau_U \in T^h_U} \tau_U$. $\tau_U$ and $\tau'_U$ have either only one common vertex or a whole edge or face if $\tau_U$ and $\tau'_U$ are in $T^h_U$. For simplicity, we again assume that $\Omega^h_U$ is a convex polygon so that $\Omega^h_U = \Omega^h_U$.

Associated with $T^h_U$ is another finite dimensional subspace $U^h$ of $L^2(\Omega^h_U)$, such that $\chi_{\tau_U}$ are polynomials of m-order ($m \geq 0$) for all $\chi \in U^h$ and $\tau_U \in T^h_U$. Here there is no requirement for continuity, and we only consider the cases $m = 0,1$ due to the lower regularity of the optimal control in general. Let $X^h = L^2(0,T; U^h)$. It is easy to see that $X^h \subset X$. Let $h_\tau (h_{\tau_U})$ denote the maximum diameter of the element $\tau$ (or $\tau_U$) in $T^h (T^h_U)$. Note that in general the sizes of the elements in $T^h_U$ are smaller than those in $T^h$ in computations. Therefore, we assume that $h_{\tau U}/h_\tau \leq C$ in this paper.

A possible semi-discrete finite element approximation of (QCP) is as follows: (QCP)$^h$

$$
\min_{u_h \in X^h, u_0 \in V^h} \left\{ \int_0^T (g(y_h) + h(u_h)) \, dt \right\},
$$

$$
\frac{\partial}{\partial t} y_h(u_h), w_h) + a(y_h(u_h), w_h) = (f + B u_h, w_h), \quad t \in (0, T], \quad \forall w_h \in V^h,
$$

$$
y_h(u_h)(x,0) = y_h^0(x), \quad x \in \Omega,
$$

where $y_h \in H^1(0,T; V^h)$, $K^h$ is a closed convex set in $U^h$, and $y_h^0 \in V^h$ is an approximation of $y_0$.

For ease of exposition, in this paper we assume that $K^h \subset K \cap U^h$. More complicated cases can be considered following the approach used in [29].

It follows that the control problem (QCP)$^h$ has a unique solution $(y_h, u_h)$ and that a pair $(y_h, u_h)$ is the solution of (QCP)$^h$ if and only if there is a co-state $p_h$ such that the triplet $(y_h, p_h, u_h)$ satisfies the following optimality conditions: (QCP-OPT)$^h$

$$
\frac{\partial}{\partial t} y_h(u_h, w_h) + a(y_h, w_h) = (f + B u_h, w_h), \quad \forall w_h \in V^h \subset V = H^1_0(\Omega),
$$

$$
y_h(x,0) = y_h^0(x), \quad x \in \Omega,
$$

$$
-(\frac{\partial}{\partial t} p_h, q_h) + a(q_h, p_h) = (g(y_h), q_h), \quad \forall q_h \in V^h \subset V = H^1_0(\Omega),
$$

$$
p_h(x,T) = 0, \quad x \in \Omega,
$$

$$
(h'(u_h) + B^* p_h, u_h - u_h) \geq 0, \quad u_h(t) \in K^h, \quad \forall u_h \in K^h \subset U^h \cap K.
$$

We now consider the fully discrete approximation for above semidiscrete problem by using the backward Euler scheme in time.

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T, \ k_i = t_i - t_{i-1}, \ i = 1, 2, \cdots, N, \ k = \max_{i \in [0,N]} \{k_i\}$. For $i = 1, 2, \cdots, N$, construct the finite element spaces $V_i^h \in H^1_0(\Omega)$ (similar as $V^h$) with the mesh $T_i^h$. 
Similarly, construct the finite element spaces $U^h_i \in L^2(\Omega)$ (similar as $U^h$) with the mesh $(T^h_i)$. Let $h_r(\theta^h_i)$ denote the maximum diameter of the element $r^h(\theta^h_i)$ in $T^h_i$ ($T^h_i$). Define mesh functions $\tau(\cdot), \tau_r(\cdot)$ and mesh size functions $h_r(\cdot), h_r(\cdot)$ such that $\tau(t) = \tau^h_i, \tau_r(t) = \tau^h_i, h_r(t) = h^h_i, h_r(t) = h^h_i$. For ease of exposition, we shall denote $\tau(t), \tau_r(t), h_r(t)$ and $h_r(t)$ by $\tau, \tau_r, h_r$ and $h_r$, respectively. Let $K^h_i \subset U^h_i \cap K$. The fully discrete approximation scheme (QCP) has is to find $(\tilde{y}^h_i, u^h_i) \in V^h_i \times K^h_i, i = 1, 2, \ldots, N$, such that

$$
\min_{\tilde{y}^h_i \in K^h_i} \{ \sum_{i=1}^{N} k_i (g(\tilde{y}^h_i) + h(u^h_i)) \},
$$

$$
\frac{(y^h_i - y^{h-1}_i, w_h)}{k_i} + a(y^h_i, w_h) = (f(x, t_i) + Bu^h_i, w_h),
$$

$$
\forall w_h \in V^h_i \subset V = H^0_0(\Omega), i = 1, \ldots, N,
$$

$$
y^h_0(x) = y^h_0(x), \quad x \in \Omega,
$$

It follows that the control problem (QCP) has a unique solution $(Y^h_i, U^h_i), i = 1, 2, \ldots, N$, and that a pair $(Y^h_i, U^h_i) \in V^h_i \times K^h_i, i = 1, 2, \ldots, N$, is the solution of (QCP) if and only if there is a co-state $P^h_i \in V^h_i, i = 1, 2, \ldots, N$, such that the triplet $(Y^h_i, P^h_i, U^h_i) \in V^h_i \times V^h_i \times K^h_i, i = 1, 2, \ldots, N$, satisfies the following optimality conditions:

$$
Y^h_i - Y^h_{i-1} = (f(x, t_i) + Bu^h_i, w_h),
$$

$$
\forall w_h \in V^h_i \subset V = H^0_0(\Omega), i = 1, \ldots, N,
$$

$$
Y^h_0(x) = y^h_0(x), \quad x \in \Omega,
$$

$$
P^h_i - P^h_{i-1} = (g(Y^h_i), q_h),
$$

$$
\forall q_h \in V^h_i \subset V = H^0_0(\Omega), i = N, \ldots, 1,
$$

$$
P^h_1 = 0, \quad x \in \Omega,
$$

$$
(h^r(U^h_i) + B^r P^h_i, v_h - U^h_i)_\Omega \geq 0 \quad \forall v_h \in K^h_i \subset U^h_i \cap K, i = 1, 2, \ldots, N.
$$

For $i = 1, 2, \ldots, N$, let

$$
Y^h_i_{|_{t_{i-1}, t_i}} = ((t_i - t) Y^h_{i-1} + (t - t_{i-1}) Y^h_i)/k_i,
$$

$$
P^h_i_{|_{t_{i-1}, t_i}} = ((t_i - t) P^h_{i-1} + (t - t_{i-1}) P^h_i)/k_i,
$$

$$
U^h_i_{|_{t_{i-1}, t_i}} = U^h_i.
$$

For any function $w \in C(0, T; L^2(\Omega))$, let $\tilde{w}(x, t)_{|_{t_{i-1}, t_i}} = w(x, t_i)$, $\tilde{w}(x, t)_{|_{t_{i-1}, t_i}} = w(x, t_i)$. Then the optimality conditions (2.6), (2.7), and (2.8) can be restated as

$$
\left( \frac{\partial Y^h_i}{\partial t}, w_h \right) + a(\tilde{Y^h}_i, w_h) = (\tilde{f} + B\tilde{U^h}_i, w_h),
$$

$$
\forall w_h \in V^h_i \subset V = H^0_0(\Omega), t \in (t_{i-1}, t_i), i = 1, 2, \ldots, N,
$$

$$
Y^h_0(x) = y^h_0(x), \quad x \in \Omega,
$$

$$
-(\partial P^h_i/\partial t, q_h) + a(q_h, \tilde{P^h}_i) = (g(\tilde{Y^h}_i), q_h),
$$

$$
\forall q_h \in V^h_i \subset V = H^0_0(\Omega), t \in (t_{i-1}, t_i), i = N, \ldots, 2, 1,
$$

$$
(h^r(U^h_i) + B^r P^h_i, v_h - U^h_i)_\Omega \geq 0 \quad \forall v_h \in K^h_i \subset U^h_i \cap K, i = 1, 2, \ldots, N.$$
\[ P_h(x, T) = 0, \ x \in \Omega. \]

\[ (h'(U_h) + B^*P_h, v_h - U_h)_V \geq 0, \ U_h \in K_i^h \subset U_i^h \cap K, \]

\[ \forall v_h \in K_i^h, \ t \in (t_{i-1}, t_i], \ i = 1, 2, \cdots, N. \]

We now describe our new scheme. We first give the discretization of the state. Let \( \Omega^h \) be a polygonal approximation to \( \Omega \) with the boundary \( \partial \Omega^h \), \( Q = \Omega \times (0, T) \), and \( Q^h = \Omega^h \times (0, T) \). We consider \((n+1)\)-simplex Lagrange elements on \( Q^h \). Let \( T^h \) be a partitioning of \( Q^h \) into disjoint regular \((n+1)\)-simplex \( \tau \), so that \( \bar{Q}^h = \bigcup_{\tau \in T^h} \bar{\tau} \). Each element has at most one face on \( \partial Q^h \), and \( \bar{\tau} \) and \( \bar{\tau}' \) have either only one common vertex or a whole edge or face if \( \tau \) and \( \tau' \in T^h \). We further require that \( P_i \in \partial Q^h \) implies \( P_i \in \partial Q \) where \( \{P_i\} \) \((i = 1, 2, \cdots, J)\) is the vertex set associated with the triangulation \( T^h \). For simplicity, we assume that \( \Omega \) is a convex polygon so that \( \Omega^h = \Omega \) and \( Q^h = Q \).

Associated with \( T^h \) is a finite dimensional subspace \( S^h \) of \( C(\bar{Q}^h) \), such that \( w|_\tau \) are polynomials of \( m \)-order \((m \geq 1)\) for all \( w \in W^h \) and \( \tau \in T^h \). Let \( W^h = S^h \cap L^2(0, T; H_0^1(\Omega)) \) and \( h_r \) denote the diameter of the element \( \tau \) in \( T^h \).

The partitioning of the domain \( Q_V = \Omega_V \times (0, T) \) is the same as that of \( \Omega \) but with the notations \( T^h, \tau_V, Q^h_V = \Omega_V \times (0, T), \partial Q^h_V \), and \( h_{rV} \).

Associated with \( T^h \) is a finite dimensional subspace \( X^h \) of \( L^2(Q^h_V) \), such that \( v|_{\tau_V} \) are polynomials of \( m \)-order \((m \geq 0)\) for all \( v \in X^h \) and \( \tau_V \in T^h \). Here there is no requirement for the continuity, and we only consider the cases \( m = 0, 1 \) due to the lower regularity of the optimal control in general. Let \( h_{rV} \) denote the diameter of the element \( \tau_V \) in \( T^h \). Let \( K^h \subset (U^h \cap K) \).

The fully discrete approximation scheme is to find \((y_h, u_h)\) such that

\[ \min_{u_h \in X^h; u_h(0) \in K^h} \left\{ \int_0^T (g(y_h) + h(u_h)) \, dt \right\}, \]

\[ \int_0^T ((\frac{\partial y_h}{\partial t}), w) + a(y_h, w)) \, dt + (y_h(0) - y_0, w(0)) = \int_0^T (f + B u_h, w) \, dt, \quad \forall w \in W^h. \]

We note that for any given \( f \) and \( u_h \), (2.13) has a unique solution since the corresponding homogenous system has only a zero solution. It follows that the control problem (5.9) has a unique solution \((y_h, u_h)\), and that a pair \((y_h, u_h)\) is the solutions of (5.9) if and only if there is co-state \( p_h \in W^h \) such that the triplet \((y_h, p_h, u_h)\) satisfies the following optimality conditions:

\[ \int_0^T ((\frac{\partial y_h}{\partial t}), w) + a(y_h, w)) \, dt + (y_h(0) - y_0, w(0)) = \int_0^T (f + B u_h, w) \, dt, \quad \forall w \in W^h, \]

\[ \int_0^T (- (\frac{\partial p_h}{\partial t}, q) + a(q, p_h)) \, dt + (p_h(T), q(T)) = \int_0^T (g'(y_h), q) \, dt, \quad \forall q \in W^h, \]

\[ \int_0^T (h'(u_h) + B^* p_h, v - u_h)_V \, dt \geq 0, \quad \forall v(t) \in K^h, v \in X^h. \]

In the following section we shall derive some \textit{a posteriori} error estimates for the finite element approximation of the optimal control problem governed by parabolic equations, which can be used as error indicators in developing adaptive finite element schemes of the control problem.
3. A posteriori Error Estimates for Backward Euler Scheme

In this paper, we only consider the case where the constraint set is of obstacle type, which is met most frequently in real applications. We are able to derive a posteriori error estimates for the finite element approximation to the parabolic optimal control problem using the backward Euler scheme.

We assume that the constraint on the control is an obstacle such that

\[ K = \{ v \in X : v \geq \alpha, \ \text{a.e. in } \Omega_U \times (0,T) \}, \]

where \( \alpha \) is a constant. We define the coincidence set (contact set) \( \Omega_U^-(t) \) and the non-coincidence set (non-contact set) \( \Omega_U^+(t) \) as follows:

\[ \Omega_U^-(t) := \{ x \in \Omega_U : u(x,t) = \alpha \}, \quad \Omega_U^+(t) := \{ x \in \Omega_U : u(x,t) > \alpha \}. \]

Let

\[ K^h_i = \{ v \in U^h_i : v \geq \alpha \text{ in } \Omega_U \times (t_{i-1}, t_i) \}, \]

\[ K^h = \{ v : v_{\mid(t_{i-1}, t_i)} \in K^h_i \}. \]

Hence, we have that \( K^h \subset K \). Hereafter, we assume that

\[ h(u) = \int_{\Omega_U} j(u), \]

where \( j(\cdot) \) is a convex continuously differentiable function on \( \mathbb{R} \). Then, it is easy to see that

\[ \langle h'(u), v \rangle_U = \langle j'(u), v \rangle_U = \int_{\Omega_U} j'(u)v. \]

We shall assume the following convexity conditions:

\[ (j'(t) - j'(s))(t - s) \geq c(t - s)^2, \ \forall s, t \in \mathbb{R}. \]

\[ (g'(v) - g'(w), v - w) \geq 0, \ \forall v, w \in Y, \]

Also, we assume that

\[ \| (Bu, w) \| = \| (v, B^*w) \| \leq C\| v \|_{L^0} \| w \|_{L^0}, \ \forall v \in U, w \in Y. \]

It can be seen that the inequality in (2.2) is now equivalent to the followings:

\[ j'(u) + B^*p \geq 0, \ u \geq \alpha, \ (j'(u) + B^*p)(u - \alpha) = 0, \ \text{a.e. in } \Omega_U \times (0,T). \]

In order to derive sharp a posteriori error estimates, we divide \( \Omega_U \) into the following three subsets:

\[ \Omega^+_\alpha = \{ x \in \Omega_U : (B^*p_h)(x, t_{i-1}) \leq -j'(\alpha) \}, \]

\[ \Omega^*_h = \{ x \in \Omega_U : (B^*p_h)(x, t_{i-1}) > -j'(\alpha), U^*_h = \alpha \}, \]

\[ \Omega^-_h = \{ x \in \Omega_U : (B^*p_h)(x, t_{i-1}) > -j'(\alpha), U^-_h > \alpha \}. \]

Then, it is easy to see that above three subsets are not overlapped each other, and

\[ \Omega_U = \Omega^-_\alpha \cup \Omega^*_h \cup \Omega^-_h, \ \text{for } i = 1, 2, \ldots, N. \]
Lemma 3.1. Let \((y, p, u)\) and \((Y_h, P_h, U_h)\) be the solutions of \((2.2)\) and \((2.9)-(2.11)\), respectively. Assume that \(j'()\) and \(j()\) are locally Lipschitz continuous. Then we have that
\[
\|u - U_h\|_{L^2(0, T; L^2(\Omega_h))}^2 \leq C \eta_h^2 + C \|\hat{P}_h - p(U_h)\|_{L^2(0, T; L^2(\Omega))}^2,
\]
where
\[
\eta_h^2 = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\Omega_h} (j'(U_h) + B^* \hat{P}_h)^2,
\]
and \(p(U_h)\) is defined by the following system:
\[
(\frac{\partial}{\partial t}y(U_h), w) + a(y(U_h), w) = (f + BU_h, w), \quad \forall w \in V,
\]
\[
y(U_h)(x, 0) = y_0(x), \quad x \in \Omega,
\]
\[
-\frac{\partial}{\partial t}p(U_h), q) + a(q, p(U_h)) = (g'(y(U_h)), q), \quad \forall q \in V,
\]
\[
p(U_h)(x, T) = 0, \quad x \in \Omega.
\]

Proof. From the uniform convexity of \(j\), we have that
\[
c\|u - U_h\|_{L^2(0, T; L^2(\Omega_h))}^2 \leq \int_0^T (j'(u) - j'(U_h), u - U_h)U
\]
\[
= \int_0^T (j'(u) + B^* p_h, u - U_h)U + \int_0^T (j'(U_h) + B^* \hat{P}_h, U_h - u)U
\]
\[
+ \int_0^T (B^*(\hat{P}_h - p(U_h)), u - U_h)U + \int_0^T (B^*(p(U_h) - p), u - U_h)U
\]
\[
= \int_0^T (j'(u) + B^* p_h, u - U_h)U + \int_0^T (j'(U_h) + B^* \hat{P}_h, U_h - u)U
\]
\[
+ \int_0^T (B^*(\hat{P}_h - p(U_h)), u - U_h)U + \int_0^T (B^*p(U_h) - p), u - U_h)U
\]
\[
\leq \int_0^T (j'(u) + B^* p_h, u - U_h)U + \int_0^T (j'(U_h) + B^* \hat{P}_h, U_h - u)U
\]
\[
+ \int_0^T (B^*(\hat{P}_h - p(U_h)), u - U_h)U := \sum_{i=1}^{3} I_i.
\]

We first estimate \(I_1\). Note that \(U_h \in K^h \subset K\). It follows from \((2.2)\) that
\[
I_1 = \int_0^T (j'(u) + B^* p_h, u - U_h)U \leq 0.
\]

Next we estimate \(I_2\). It is clear that for any \(t \in (t_{i-1}, t_i]\),
\[
(j'(U_h) + B^* \hat{P}_h, U_h - u)U
\]
\[
= \int_{t_{i-1}}^{t_i} (j'(U_h) + B^* \hat{P}_h)(U_h - u)U + \int_{t_{i-1}}^{t_i} (j'(U_h) + B^* \hat{P}_h)(U_h - u)
\]
\[
+ \int_{t_{i-1}}^{t_i} (j'(\alpha) + B^* \hat{P}_h)(\alpha - u).
\]

(3.13)
Firstly it is easy to see that
\[
\int_{\Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)(U_h - u) 
\leq C \int_{\Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)^2 + C\delta ||U_h - u||^2_{L^2(\Omega_u)}.
\]

(3.14)

Secondly, let \( \tau_U \) be such that \( U_h^\tau|_{\tau_U} > \alpha \). It follows from (2.11) that there exist \( \epsilon > 0 \) and \( \psi \in U_h^\epsilon \), such that \( \psi \geq 0 \), \( ||\psi||_{L^\infty(\tau_U)} = 1 \) and \( U_h - \epsilon \psi \in K^h \). Hence,
\[
\int_{\tau_U} (j'(U_h) + B^* \tilde{P}_h)(U_h - (U_h - \epsilon \psi)) = \epsilon \int_{\tau_U} (j'(U_h) + B^* \tilde{P}_h)\psi \leq 0.
\]

Note that on \( \Omega^+_u \), \( (j'(U_h) + B^* \tilde{P}_h) > (j'(\alpha) + B^* \tilde{P}_h) > 0 \). We have that
\[
\int_{\tau_U \cap \Omega^+_u} |j'(U_h) + B^* \tilde{P}_h|^2 = \int_{\tau_U \cap \Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)\psi 
\leq \int_{\tau_U \cap \Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)^2 \leq \int_{\tau_U \cap \Omega^+_u} |j'(U_h) + B^* \tilde{P}_h|^2.
\]

Let \( \hat{\tau}_U \) be the reference element of \( \tau_U \), \( \tau^0_U = \tau_U \cap \Omega^+_u \), and \( \tau^0_U \subset \hat{\tau}_U \) be the image of \( \tau^0_U \). Let \( n \) be the dimension of \( \Omega_U \). Note that \( j'(\cdot) \) is locally Lipschitz continuous. It follows from the equivalence of the norm in a finite dimensional space that
\[
\int_{\tau^0_U} |j'(U_h) + B^* \tilde{P}_h|^2 \leq Ch^n \int_{\tau^0_U} |j'(U_h) + B^* \tilde{P}_h|^2 
\leq Ch^n (\int_{\tau^0_U} |j'(U_h) + B^* \tilde{P}_h|^2)^2 \leq Ch^n (\int_{\tau^0_U} |j'(U_h) + B^* \tilde{P}_h|^2)^2 
\leq C \int_{\tau_U \cap \Omega^+_u} |j'(U_h) + B^* \tilde{P}_h|^2.
\]

Therefore,
\[
\int_{\Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)(U_h - u) 
\leq C \int_{\Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)^2 + C\delta ||U_h - u||^2_{L^2(\Omega_u)} 
\leq C \int_{\Omega^+_u} (j'(U_h) + B^* \tilde{P}_h)^2 + C\delta ||U_h - u||^2_{L^2(\Omega_u)}.
\]

(3.15)

It follows from the definition of \( \Omega^+_u \) that \( (j'(\alpha) + B^* \tilde{P}_h) > 0 \) on \( \Omega^+_u \). Then, we have
\[
\int_{\Omega^+_u} (j'(\alpha) + B^* \tilde{P}_h)((\alpha - u) \leq 0.
\]

(3.16)

Thus it follows from (3.13)-(3.16) that
\[
I_2 = \int_0^T (j'(U_h) + B^* \tilde{P}_h,U_h - u) = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (j'(U_h) + B^* \tilde{P}_h,U_h - u)
\leq C \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (j'(U_h) + B^* \tilde{P}_h)^2 + C\delta \sum_{i=1}^N \int_{t_{i-1}}^{t_i} ||U_h - u||^2_{L^2(\Omega_u)} 
= C\eta^2 + C\delta ||U_h - u||^2_{L^2(\Omega_u)}.
\]

(3.17)
Finally for $I_3$, it is easy to show that
\[
I_3 = \int_0^T (B^* (\hat{P}_h - p(U_h)), u - U_h) v
\]
(3.18)
\[
\leq C \int_0^T \|B^* (\hat{P}_h - p(U_h))\|_{L^2(\Omega)}^2 + C \delta \int_0^T \|U_h - u\|_{L^2(\Omega)}^2
\]
\[
\leq C\|\hat{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 + C\|U_h - u\|_{L^2(0,T;L^2(\Omega))}^2.
\]
Thus, we obtain from (3.11), (3.12), (3.17) and (3.18) that
\[
\|U_h - u\|_{L^2(0,T;L^2(\Omega))}^2 \leq C\eta_h^2 + C\|\hat{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2.
\]
This proves (3.8).

The following lemmas are important in deriving a posteriori error estimates of residual type.

**Lemma 3.2.** [9] Let $\pi_h$ be standard Lagrange interpolation operator. For $m = 0$ or $1$, $q > \frac{m}{2}$ and $v \in W^{2,q}(\Omega)$,
\[
\|v - \pi_h v\|_{W^{m,q}(\Omega)} \leq Ch^{2-m}\|v\|_{W^{2,q}(\Omega)}.
\]

**Lemma 3.3.** [23] For all $v \in W^{1,q}(\Omega')$, $1 \leq q < \infty$,
\[
\|v\|_{W^{1,q}(\Omega')} \leq C(h^{-\frac{1}{q}}\|v\|_{W^{1,q}(\Omega)} + h^{1-\frac{1}{q}}\|v\|_{W^{1,q}(\Omega)}).
\]

In order to estimate the error $\|\hat{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}$, we shall use the following two equations: For given $F \in L^2(0,T;L^2(\Omega))$,
\[
\begin{cases}
\frac{\partial \phi}{\partial t} - \text{div}(A \nabla \phi) = F, & (x, t) \in \Omega \times (0,T], \\
\phi|_{\partial \Omega} = 0, & t \in [0,T], \\
\phi(x,0) = 0, & x \in \Omega,
\end{cases}
\]
(3.19)
\[
\begin{cases}
- \frac{\partial \psi}{\partial t} - \text{div}(A^* \nabla \psi) = F, & (x, t) \in \Omega \times [0,T], \\
\psi|_{\partial \Omega} = 0, & t \in [0,T], \\
\psi(x,T) = 0, & x \in \Omega.
\end{cases}
\]
(3.20)

The following well known stability results are presented in [21].

**Lemma 3.4.** [21] Assume that $\Omega$ is a convex domain. Let $\phi$ and $\psi$ be the solutions of (3.19) and (3.20), respectively. Then, for $v = \phi$ or $v = \psi$,
\[
\|v\|_{L^2(0,T;L^2(\Omega))} \leq C\|F\|_{L^2(0,T;L^2(\Omega))},
\]
\[
\|\nabla v\|_{L^2(0,T;L^2(\Omega))} \leq C\|F\|_{L^2(0,T;L^2(\Omega))},
\]
\[
\|D^2 v\|_{L^2(0,T;L^2(\Omega))} \leq C\|F\|_{L^2(0,T;L^2(\Omega))},
\]
\[
\|\frac{\partial v}{\partial t}\|_{L^2(0,T;L^2(\Omega))} \leq C\|F\|_{L^2(0,T;L^2(\Omega))},
\]

where $D^2 v = \frac{\partial^2 v}{\partial x_i \partial x_j}, 1 \leq i, j \leq n$.

Now, we are in the position of estimating the error $\|\hat{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}$.

**Lemma 3.5.** Assume that the domain $\Omega$ is convex. Let $(Y_h, P_h, U_h)$ be the solutions of (2.9)-(2.11), let $(y(U_h), p(U_h))$ be defined by (3.9) and (3.10). Then,
\[
\|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 + \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \leq C\sum_{i=1}^{10} \eta_i^2,
\]
(3.21)
where

\[ v_2^2 = \int_0^T \sum_l h_t^2 \left( \int_\tau \frac{\partial P_h}{\partial t} + \text{div}(A^* \nabla \tilde{P}_h) + g'(\tilde{Y}_h))^2 \right) dt \]

\[ v_3^2 = \int_0^T \sum_l \int_\tau h_t^2 [A^* \nabla \tilde{P}_h \cdot n]^2 dl dt \]

\[ v_k^2 = ||g(\tilde{Y}_h) - g(\tilde{Y}_h))||_{L^2(0,T;L^2(\Omega))}^2 \]

\[ v_5^2 = \int_0^T \int_\Omega |A^* \nabla (P_h - \tilde{P}_h)|^2 dx dt \]

\[ v_6^2 = \int_0^T \sum_l h_t^2 \left( \int_\tau \frac{\partial \tilde{Y}_h}{\partial t} - \text{div}(A \nabla \tilde{Y}_h) - \tilde{f} - BU_h)^2 \right) dt \]

\[ v_7^2 = \int_0^T \sum_l h_t^2 [A \nabla \tilde{Y}_h \cdot n]^2 dl dt \]

\[ v_8^2 = ||\tilde{f} - f||_{L^2(0,T;L^2(\Omega))}^2 \]

\[ v_9^2 = \int_0^T \int_\Omega |A \nabla (\tilde{Y}_h - \tilde{Y}_h)|^2 dx dt \]

\[ \eta_0^2 = ||(\tilde{Y}_h(x,0) - y_0(x))||_{L^2(\Omega)}, \]

where \( h_t \) is the size of the face \( l \), \([A \nabla \tilde{Y}_h \cdot n]_l \) and \([A^* \nabla \tilde{P}_h \cdot n]_l \) are the \( A^- \) normal and \( A^*^- \) normal derivative jumps over the interior face \( l \) respectively, defined by

\[ [A \nabla \tilde{Y}_h \cdot n]_l = (A \nabla \tilde{Y}_h|_{\tau^+} - A \nabla \tilde{Y}_h|_{\tau^-}) \cdot n, \]

\[ [A^* \nabla \tilde{P}_h \cdot n]_l = (A^* \nabla \tilde{P}_h|_{\tau^+} - A^* \nabla \tilde{P}_h|_{\tau^-}) \cdot n, \]

where \( n \) is the unit normal vector on \( l = \tau^+_l \cap \tau^-_l \) outwards \( \tau^-_l \). For later convenience, we define \([A \nabla \tilde{Y}_h \cdot n]_l = 0 \) and \([A^* \nabla \tilde{P}_h \cdot n]_l = 0 \) when \( l \subset \partial \Omega \).

Proof. Let \( \phi \) be the solution of (3.19) with \( F = P_h - p(U_h) \). Let \( \phi_I = \pi_h \phi \) be the interpolation of \( \phi \) defined as in Lemma 3.2. Then it follows from (3.10) and (2.10) that

\[ ||P_h - p(U_h)||_{L^2(\Omega)}^2 = \int_0^T (P_h - p(U_h), F) dt \]

\[ = \int_0^T \left( - \left( \frac{\partial}{\partial t} (P_h - p(U_h)), \phi \right) + a(\phi, P_h - p(U_h)) \right) dt \]

\[ = \int_0^T \left( - \left( \frac{\partial}{\partial t} (P_h - p(U_h)), \phi - \phi_I \right) + a(\phi - \phi_I, \tilde{P}_h - p(U_h)) \right) dt \]

\[ + \int_0^T \left( - \left( \frac{\partial}{\partial t} (P_h - p(U_h)), \phi_I \right) + a(\phi_I, \tilde{P}_h - p(U_h)) \right) dt \]

\[ + \int_0^T a(\phi, P_h - \tilde{P}_h) dt \]
\[
(3.22) \quad = \int_0^T \left( -\frac{\partial P_h}{\partial t} - g'(y(U_h)), \phi - \phi_t \right) dt + \int_0^T a(\phi - \phi_t, \tilde{P}_h) dt \\
+ \int_0^T (g'(\tilde{Y}_h) - g'(y(U_h)), \phi_t) dt + \int_0^T a(\phi, P_h - \tilde{P}_h) dt \\
= \int_0^T \left( -\frac{\partial P_h}{\partial t} - \text{div}(A^*\nabla \tilde{P}_h) - g'(\tilde{Y}_h), \phi - \phi_t \right) dt \\
+ \int_0^T \sum_r \int_{\partial r} (A^*\nabla \tilde{P}_h \cdot n)(\phi - \phi_t) d\Gamma dt \\
+ \int_0^T (g'(\tilde{Y}_h) - g'(y(U_h)), \phi) dt + \int_0^T a(\phi, P_h - \tilde{P}_h) dt \\
= I_1 + I_2 + I_3 + I_4.
\]

It follows from Lemma 3.2 and Lemma 3.4 that
\[
I_1 = \int_0^T \left( -\frac{\partial P_h}{\partial t} - \text{div}(A^*\nabla \tilde{P}_h) - g'(\tilde{Y}_h), \phi - \phi_t \right) dt \\
\leq C \int_0^T \sum_r \left\| \frac{\partial P_h}{\partial t} + \text{div}(A^*\nabla \tilde{P}_h) + g'(\tilde{Y}_h) \right\|_{L^2(\Omega)} \left\| \phi \right\|_{H^2(\Omega)} dt \\
\leq C \int_0^T \sum_r h_r^2 \int_{\Gamma_r} \left( \frac{\partial P_h}{\partial t} + \text{div}(A^*\nabla \tilde{P}_h) + g'(\tilde{Y}_h) \right)^2 dz dt \\
+ C\delta \int_0^T \left\| \phi \right\|_{L^2(\Omega)}^2 dz dt \\
\leq C\eta_h^2 + C\delta \| P_h - p(U_h) \|_{L^2(0,T;L^2(\Omega))}^2.
\]

Similarly, it follows from Lemmas 3.2, 3.3 and 3.4 that
\[
I_2 = \int_0^T \sum_r \int_{\partial r} (A^*\nabla \tilde{P}_h \cdot n)(\phi - \phi_t) d\Gamma dt \\
= \int_0^T \sum_r \int_{\partial r} [A^*\nabla \tilde{P}_h \cdot n](\phi - \phi_t) d\Gamma dt \\
\leq C \int_0^T \sum_r \int_{\Gamma_r} h_r^2 \left\| A^*\nabla \tilde{P}_h \cdot n \right\|^2_{L^2(\Omega)} dz dt \\
+ C\delta \int_0^T \left\| \phi \right\|_{L^2(\Omega)}^2 dz dt \\
\leq C\eta_h^2 + C\delta \| P_h - p(U_h) \|_{L^2(0,T;L^2(\Omega))}^2.
\]

It follows from Lemma 3.4 and Schwartz inequality that
\[
I_3 = \int_0^T (g'(\tilde{Y}_h) - g'(y(U_h)), \phi) dt \\
\leq C \| g'(\tilde{Y}_h) - g'(y(U_h)) \|_{L^2(0,T;L^2(\Omega))} + C\delta \int_0^T \| \phi(x,t) \|_{L^2(\Omega)}^2 \\
\leq C \| g'(\tilde{Y}_h) - g'(y(U_h)) \|_{L^2(0,T;L^2(\Omega))} + C\eta_h^2 + C\| \phi(x,t) \|_{L^2(0,T;L^2(\Omega))}^2 \\
\leq C \| Y_h - y(U_h) \|_{L^2(0,T;L^2(\Omega))}^2 + C\delta \| P_h - p(U_h) \|_{L^2(0,T;L^2(\Omega))}^2,
\]
\[ I_4 = \int_0^T a(\phi, P_h - \tilde{P}_h) \, dt \]

\[ \leq C \int_0^T \int_{\Omega} |A^* \nabla (P_h - \tilde{P}_h)|^2 \, dx \, dt + C \delta \int_0^T \int_{\Omega} |\nabla \phi|^2 \, dx \, dt \]

\[ \leq C \eta^2 + C \delta \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}. \]

Then it follows from (3.22)-(3.26) that

\[ \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))} \leq C \sum_{i=2}^5 \eta_i^2 + C \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}. \]

Similarly, let \( \psi \) be the solution of (3.20) with \( F = Y_h - y(U_h) \). We have that

\[ \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))} = \int_0^T (Y_h - y(U_h), F) \]

\[ = \int_0^T \left( \left( \frac{\partial}{\partial t} (Y_h - y(U_h)), \psi \right) + a(Y_h - y(U_h), \psi) \right) \, dt 
+ (Y_h - y(U_h))(x,0), \psi(x,0) \right) dt 
+ \int_0^T \left( \left( \frac{\partial}{\partial t} (Y_h - y(U_h)), \psi_1 \right) + a(Y_h - y(U_h), \psi_1) \right) \, dt 
+ \int_0^T a(Y_h - \tilde{Y}_h, \psi) \, dt + ((Y_h - y(U_h))(x,0), \psi(x,0)) \right) dt 
+ \int_0^T \left( \left( \frac{\partial Y_h}{\partial t} - f - BU_h, \psi \right) \right) \right) \, dt + \int_0^T (\tilde{f} - f, \psi_1) \, dt 
+ \int_0^T a(Y_h - \tilde{Y}_h, \psi) \, dt + ((Y_h - y(U_h))(x,0), \psi(x,0)) \right) dt 
+ \int_0^T \left( \left( \frac{\partial \tilde{Y}_h}{\partial t} - \text{div}(A \nabla \tilde{Y}_h) - \tilde{f} - BU_h, \psi \right) \right) \, dt 
+ \int_0^T \sum_r \int_{\partial r} (A \nabla \tilde{Y}_h \cdot n)(\psi - \psi_1) \, d\Gamma \right) \, dt + \int_0^T (\tilde{f} - f, \psi) \, dt 
+ \int_0^T a(Y_h - \tilde{Y}_h, \psi) \, dt + (\tilde{Y}_h(x,0) - y_0(x), \psi(x,0)) \right) \]
\[
\leq C \int_0^T \sum_i h_i^4 \int_\Omega \left( \frac{\partial Y_h}{\partial t} - \text{div}(A\nabla Y_h) - \tilde{f} - BU_h \right)^2 dx dt \\
+ C \int_0^T \sum_i h_i^2 \int_\Omega [A\nabla (Y_h - \bar{Y}_h)]^2 dx dt + C \|f\|_{L^2(0,T;L^2(\Omega))}^2 \\
+ C \int_0^T \int_\Omega |A\nabla (Y_h - \bar{Y}_h)|^2 dx dt + C\|Y_h(x, 0) - y_0(x)\|_{L^2(\Omega)}^2 \\
+ C\delta \int_{\Omega} \|\psi\|_{L^2(\Omega)}^2 + C\delta\|\psi(x, 0)\|_{H^1(\Omega)}^2 \\
\leq C \sum_{i=6}^{10} \eta_i^2 + C\delta\|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2.
\]

Hence,

(3.28) \quad \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=6}^{10} \eta_i^2.

Then, (3.21) follows from (3.27) and (3.28).

**Theorem 3.1.** Let \((y, p, u)\) and \((Y_h, P_h, U_h)\) be the solutions of (2.2) and (2.9)–(2.11), respectively. Assume that all the conditions in Lemma 3.1 and 3.5 are valid. Then

(3.29) \quad \|Y_h - y\|_{L^2(0,T;L^2(\Omega))}^2 + \|P_h - p\|_{L^2(0,T;L^2(\Omega))}^2 + \|U_h - u\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=1}^{10} \eta_i^2,

where \(\eta_i\) is defined in Lemma 3.1, \(\eta_i, i = 2, \cdots, 10\), are defined in Lemma 3.5.

Proof. It follows from Lemma 3.1 and 3.5 that

\[
\|u - U_h\|_{L^2(0,T;L^2(\Omega))}^2 \leq C\eta_i^2 + C\|\bar{P}_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \\
\leq C\eta_i^2 + C\|\bar{P}_h - P_h\|_{L^2(0,T;L^2(\Omega))}^2 + C\|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))}^2 \\
\leq C\eta_i^2 + C\|\bar{P}_h - P_h\|_{L^2(0,T;L^2(\Omega))}^2 + C \sum_{i=2}^{10} \eta_i^2.
\]

Note that \(A\) is positive definite. It follows from Poincare's inequality that

\[
\|\bar{P}_h - P_h\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \int_0^T \int_\Omega |\nabla (\bar{P}_h - P_h)|^2 dx dt \\
\leq C \int_0^T \int_\Omega |A^{\ast}\nabla (\bar{P}_h - P_h)|^2 dx dt = C\eta_i^2.
\]

Then, it follows from (3.30) and (3.31) that

(3.32) \quad \|u - U_h\|_{L^2(0,T;L^2(\Omega))}^2 \leq C \sum_{i=2}^{10} \eta_i^2.

Note that

\[
\|Y_h - y\|_{L^2(0,T;L^2(\Omega))} \leq \|Y_h - y(U_h)\|_{L^2(0,T;L^2(\Omega))} + \|y(U_h) - y\|_{L^2(0,T;L^2(\Omega))},
\]

\[
\|P_h - p\|_{L^2(0,T;L^2(\Omega))} \leq \|P_h - p(U_h)\|_{L^2(0,T;L^2(\Omega))} + \|p(U_h) - p\|_{L^2(0,T;L^2(\Omega))},
\]

and

\[
\|y(U_h) - y\|_{L^2(0,T;L^2(\Omega))} \leq C\|U_h - u\|_{L^2(0,T;L^2(\Omega))},
\]
\[ \| p(U_h) - p \|_{L^2(0,T;L^2(\Omega))} \leq C \| y(U_h) - y \|_{L^2(0,T;L^2(\Omega))} \leq C \| U_h - u \|_{L^2(0,T;L^2(\Omega))}. \]

Then, it follows from Lemma 3.5 and (3.3) that
\[ \| Y_h - y \|_{L^2(0,T;L^2(\Omega))} + \| P_h - p \|_{L^2(0,T;L^2(\Omega))} \leq C \sum_{i=1}^{10} \eta_i^2 + C \| U_h - u \|_{L^2(0,T;L^2(\Omega))} \]
(3.33)
Therefore, (3.29) follows from (3.32) and (3.33).

4. A posteriori Error Estimates for the New Scheme

In this section, we derive error estimates for the new finite element approximation scheme of the parabolic optimal control problem (2.14) with the constant obstacle constraint.

We assume that the constraint on the control is the same as in (3.1). We define the coincidence set (contact set) \( Q^-_U \) and the noncoincidence set (non-contact set) \( Q^+_U \) as follows:
\[ Q^-_U := \{ (x,t) \in Q_U : u(x,t) = \alpha \}, \quad Q^+_U := \{ (x,t) \in Q_U : u(x,t) > \alpha \}. \]

Let
(4.1)
\[ K^h = \{ v \in X^h : v \geq \alpha \text{ a.e. in } Q_U \} \subset K. \]

Similarly we divide \( Q_U \) into the following three subsets:
\[ Q^-_U = \{ (x,t) \in Q_U : (B^*p_h)(x,t) \leq -j'(\alpha) \}, \]
\[ Q^0_U = \{ (x,t) \in Q_U : (B^*p_h)(x,t) > -j'(\alpha), u_h = \alpha \}, \]
\[ Q^+_U = \{ (x,t) \in Q_U : (B^*p_h)(x,t) > -j'(\alpha), u_h > \alpha \}. \]

Then, it is easy to see that above three subsets are not overlapped each other, and
\[ Q_U = \tilde{Q}_a^- \cup \tilde{Q}_a^0 \cup \tilde{Q}_a^+. \]

Lemma 4.1. Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of (2.2) and (2.14), respectively. Assume that the conditions (3.4)–(3.6) hold, \(K^h\) is defined in (4.1). Moreover, assume that \(f(\cdot)\) and \(g(\cdot)\) are Lipschitz continuous. Then
(4.2)
\[ \| u_h - u \|_{L^2(Q_U)} \leq C(\eta^2 + \| y_h - p_h \|_{L^2(Q_U)}), \]
where
\[ \eta^2 = \| j'(u_h) + B^*p_h \|_{L^2(Q_U)}^2, \]
and \(p^h_h\) is defined by the following system:
(4.3)
\[ \frac{\partial y_h}{\partial t}, w + a(y_h, w) = (f + Bu_h, w), \quad \forall w \in Y, \quad t \in (0,T], \quad y_h(0) = y_0, \]
(4.4)
\[ - \frac{\partial p_h}{\partial t}, q + a(q, p_h) = (g(y_h), q), \quad \forall q \in Y, \quad t \in [0,T], \quad p_h(T) = 0. \]

Proof. It follows from (2.2), (4.3), and (4.4) that
(4.5)
\[ \frac{\partial}{\partial t}(y_h - y), w + a(y_h - y, w) = (B(u_h - u), w), \quad \forall w \in Y, \]
(4.6)
\[ - \frac{\partial}{\partial t}(p_h - p), q + a(q, p_h - p) = (g(y_h) - g(y), q), \quad \forall q \in Y. \]
Taking $w = p^u - p$ in (4.5) and $q = y^u - y$ in (4.6) leads to

$$
(B(u_h - u), p^u - p)_Q = (g'(y^u) - g'(y), y^u - y)_Q \geq 0.
$$

From the uniform convexity of $h$ (3.4), (2.2), and (4.7), we have that

$$
c\|u_h - u\|^2_{L^2(Qe)} \leq (j'(u) - j'(u_h), u - u_h)_Q
$$

$$
= (j'(u) + B^*p_h, u - u_h)_Q + (j'(u) + B^*p_h, u - u_h)_Q
$$

$$
+ (B^*(p - p^u), u - u_h)_Q + (B^*(p^u - p), u - u_h)_Q
$$

$$
\leq (j'(u) + B^*p_h, u - u_h)_Q + (B^*(p^u - p), u - u_h)_Q
$$

$$
=: I_1 + I_2.
$$

We estimate $I_1$. It is clear that

$$
I_1 = \int_{Q^+} (j'(u_h) + B^*p_h)(u_h - u) + \int_{Q^-} (j'(u_h) + B^*p_h)(u_h - u)
$$

$$
+ \int_{Q^a} (j'(\alpha) + B^*p_h)(\alpha - u).
$$

Firstly, it is easy to see that

$$
\int_{Q^+} (j'(u_h) + B^*p_h)(u_h - u) \leq C\eta_i^2 + C\delta\|u_h - u\|^2_{L^2(Qe)}.
$$

Secondly, let $\tau_U$ be such that $u_h|\tau_U > \alpha$. Then it follows from (2.14) that there exist $\epsilon > 0$ and $\psi \in X^h$ such that $\psi \geq 0$, $\|\psi\|_{L^2(\tau_U)} = 1$ and

$$
\int_{\tau_U} (j'(u_h) + B^*p_h)(u_h - (u_h - \epsilon\psi)) = \epsilon \int_{\tau_U} (j'(u_h) + B^*p_h)\psi \leq 0.
$$

Noting $j'(u_h) + B^*p_h \geq j'(\alpha) + B^*p_h > 0$ on $Q^a$, we have that

$$
\int_{\tau_U \cap Q^+} j'(u_h) + B^*p_h|\psi = \int_{\tau_U \cap Q^+} (j'(u_h) + B^*p_h)\psi
$$

$$
\leq - \int_{\tau_U \cap Q^-} (j'(u_h) + B^*p_h)\psi \leq \int_{\tau_U \cap Q^-} |j'(u_h) + B^*p_h|.
$$

Let $\hat{\tau}_U$ be the reference element of $\tau_U$, $\tau_U^+ = \tau_U \cap Q^+$, and $\hat{\tau}_U^+ \subset \hat{\tau}_U$ be the image of $\tau_U^+$. Let $n$ be the dimension of $\Omega$. Note that $j'(\cdot)$ is locally Lipschitz continuous. It follows from the equivalence of the norm in a finite dimensional space that

$$
\int_{\tau_U} |j'(u_h) + B^*p_h|^2 \leq C \int_{\tau_U} |j'(u_h) + B^*p_h|^2
$$

$$
\leq C h_i^{n+1} \int_{\hat{\tau}_U^+} |j'(u_h) + B^*p_h|^2 \leq C h_i^{n-1} \int_{\hat{\tau}_U^{-1}} |j'(u_h) + B^*p_h|^2
$$

$$
\leq C \int_{\tau_U \cap Q^-} |j'(u_h) + B^*p_h|^2 \leq C \int_{\tau_U \cap Q^-} |j'(u_h) + B^*p_h|^2.
$$

Therefore,

$$
\int_{Q^+} (j'(u_h) + B^*p_h)(u_h - u) \leq C \int_{Q^+} (j'(u_h) + B^*p_h)^2 + C\delta\|u_h - u\|^2_{L^2(Qe)}
$$

$$
\leq C\eta_i^2 + C\delta\|u_h - u\|^2_{L^2(Qe)}.
$$
It follows from the definition of $Q_h^0$ that $(f(\alpha) + B^*p_h) \geq 0$ on $Q_h^0$. Then, we have

$$
(4.12) \quad \int_{Q_h^0} (f'(\alpha) + B^*p_h)(\alpha - u) \leq 0.
$$

Thus it follows from (4.9)–(4.12) that

$$
(4.13) \quad I_1 \leq C \eta_h^2 + C\delta \|u_h - u\|_{L^2(Q)}^2.
$$

For $I_2$, it is easy to show that

$$
(4.14) \quad I_2 \leq C\|B^*(p_h - p_h^m)\|_{L^2(Q)}^2 + C\delta \|u_h - u\|_{L^2(Q)}^2,
$$

Combining (4.13) and (4.14) leads to

$$
\|u_h - u\|_{L^2(Q)}^2 \leq C(\eta_h^2 + \|p_h - p_h^m\|_{L^2(Q)}^2).
$$

To estimate $\|p_h - p_h^m\|_{L^2(Q)}^2$, we need the following lemmas which are similar to Lemmas 3.2–3.3.

**Lemma 4.2.** Let $\pi_h^m$ be the average interpolation operator defined in [21]. For any $v \in W^{1,q}(Q^h)$ and $1 \leq q \leq \infty$,

$$
||v - \pi_h^m v||_{L^2(Q^h)} \leq C \sum_{\tau \cap \tau' \neq \emptyset} h_{\tau'}^m \|v\|_{m,q,\tau'}, \quad v \in W^{m,q}(\tau), \quad m = 0, 1, 2,
$$

$$
||v - \pi_h^m v||_{L^2(Q^h)} \leq C \sum_{\tau \cap \tau' \neq \emptyset} h_{\tau'}^m \|v\|_{1,q,\tau'}, \quad v \in W^{1,q}(\tau), \quad l = 1, 2.
$$

**Lemma 4.3** ([23]). For all $v \in W^{1,q}(Q^h)$, $1 \leq q \leq \infty$,

$$
(4.15) \quad \|v\|_{L^2(Q^h)} \leq C(h_{\tau}^{-1/q}\|v\|_{L^2(\tau)} + h_{\tau}^{1-1/q}\|v\|_{L^2(\tau)}).
$$

Now, we estimate the error $\|p_h - p_h^m\|_{L^2(Q)}^2$. Let $\partial T^h$ be the set consisting of all the faces $l$ of any $\tau \in T^h$ such that $l$ is not on $\partial Q$. The spacial $A$-normal derivative jump over the interior face $l$ is defined by

$$
[(A \nabla v_h) \cdot n] = ((A \nabla v_h)|_{\tau_1} - (A \nabla v_h)|_{\tau_2}) \cdot n, \quad n = (n_x, n_y),
$$

where $n$ is the unit outer normal vector of $\tau_1$ on $l = \tau_1 \cap \tau_2$, $n_x$ and $n_y$ are the spatial and time components of $n$, respectively. Let $h_l$ be the diameter of the face $l$.

**Lemma 4.4.** Assume that the domain $\Omega$ is convex and all the conditions of Lemma 4.1 hold. Let $(y, p, u)$ and $(y_h, p_h, u_h)$ be the solutions of (2.2) and (2.14), respectively. Let $(y_h^m, p_h^m)$ be defined in (4.3) and (4.4). Then,

$$
\|p_h - p_h^m\|_{L^2(Q)} \leq C \sum_{i=2}^6 \eta_i^2,
$$

where $\eta_i$ is the error in $A$-norm at level $i$. 

where
\[ v_2^2 = \sum_{\tau \in T_h} h_2^2 \left\| \frac{\partial p_h}{\partial t} + g'(y_h) + \text{div}(A^* \nabla p_h) \right\|_{0,\tau}^2, \]
\[ v_3^2 = \sum_{\tau \in T_h} h_3^2 \left\| \frac{\partial y_h}{\partial t} - f - Bu_h - \text{div}(A \nabla y_h) \right\|_{0,\tau}^2, \]
\[ \nu_4^2 = \sum_{i \in \Omega_{T_h}} h_i \left||((A^* \nabla y_h) \cdot \mathbf{n}_x)||_{0,i}^2 + \left||((A^* \nabla p_h) \cdot \mathbf{n}_x)||_{0,i}^2 \right. \]
\[ \nu_5^2 = \sum_{i \in \Omega_{T_h}} h_i |p_h|^2_{0,i}, \]
\[ \nu_6^2 = \sum_{i \in \Omega_{T_h}} h_i |y_h - y_0|^2_{0,i}. \]

Proof. Let \( \varphi \) be the solution of (3.19) with \( F = p_h - p^u_h \) and \( \varphi_I \) be the interpolation of \( \varphi \) such that \( \varphi_I = \pi_h^u \varphi \), where \( \pi_h^u \) is defined in Lemma 4.2 corresponding to the partitioning \( T_h \). Then it follows from (3.19), (4.4), (2.14), and the Green’s formula that
\[
||p_h - p^u_h||^2_{L^2(Q)} = \int_0^T (p_h - p^u_h, F) dt = \int_0^T (p_h - p^u_h, \frac{\partial \varphi}{\partial t} - \text{div}(A \nabla \varphi)) dt
\]
\[
= \int_0^T \left( - \frac{\partial}{\partial t} (p_h - p^u_h, \varphi) + a(\varphi, p_h - p^u_h) \right) dt + (p_h(T), \varphi(T))
\]
\[
= \int_0^T \left( - \frac{\partial p_h}{\partial t} + g'(y_h) \varphi + a(\varphi, p_h) \right) dt + (p_h(T), \varphi(T))
\]
\[
+ \int_0^T \left( \frac{\partial p_h}{\partial t} + g'(y_h)(\varphi - \varphi_I) - a(\varphi_I, p_h) \right) dt - (p_h(T), \varphi_T(T))
\]
\[
= \int_0^T \left( - \frac{\partial p_h}{\partial t} + g'(y_h) + \text{div}(A^* \nabla p_h), \varphi - \varphi_I \right) dt + \int_0^T \left( g'(y_h) - g'(y^u_h), \varphi \right) dt
\]
\[
+ \sum_{i \in \Omega_{T_h}} \int_{\Omega} \left|((A^* \nabla p_h) \cdot \mathbf{n}_x)\right| \left|((A^* \nabla p_h) \cdot \mathbf{n}_x)\right| + (p_h(T), \varphi(T) - \varphi_T(T))
\]
\[
:= I_1 + I_2 + I_3 + I_4.
\]

For simplicity, let
\[ r_p := \frac{\partial p_h}{\partial t} + g'(y_h) + \text{div}(A^* \nabla p_h). \]

By Lemma 4.2 and Lemma 3.4,
\[
I_1 = \int_0^T (r_p, (\pi_h^u - I) \varphi) dt \leq C \sum_{\tau \in T_h} h_2^2 ||r_p||^2_{0,\tau} + \delta ||\varphi||_{H^1(Q)}^2
\]
\[
\leq C(\hat{\nu}_2^2 + \delta ||p^u_h - p_h||^2_{L^2(Q)}).
\]

It is easy to see that
\[
I_2 = \int_0^T (g'(y_h) - g'(y^u_h), \varphi) dt \leq C ||y_h - y^u_h||^2_{L^2(Q)} + C\delta ||p^u_h - p_h||^2_{L^2(Q)}.
\]

Similarly, by Lemma 4.2, Lemma 3.3, and Lemma 3.4,
\[
I_3 = \sum_{i \in \Omega_{T_h}} \int_{\Omega} \left|((A^* \nabla p_h) \cdot \mathbf{n}_x)\right| \left|((A^* \nabla p_h) \cdot \mathbf{n}_x)\right| \leq \sum_{i \in \Omega_{T_h}} \left|((A^* \nabla p_h) \cdot \mathbf{n}_x)\right| ||\varphi - \varphi_I||_{0,i}^2
\]
\[
\leq C \sum_{i \in \Omega_{T_h}} h_i \left|((A^* \nabla p_h) \cdot \mathbf{n}_x)\right| ||\varphi - \varphi_I||_{H^1(Q)}^2
\]
\[
\leq C(\hat{\nu}_4^2 + \delta ||p^u_h - p_h||^2_{L^2(Q)}).
\]
\begin{equation}
I_4 = (p_h(T), \varphi(T) - \varphi_1(T)) \leq C \sum_{t \in \{t^k : 0 < t < T\}} h_t \|p_h\|_{0,t}^2 + \delta \|\varphi\|^2_{H^1(Q)}
\end{equation}

Thus, the above estimates give

\begin{equation}
\|p_h - p^{n_h}\|^2_{L^2(Q)} \leq C(\bar{\eta}_2^2 + \bar{\eta}_3^2 + \|p_h - p^{n_h}\|^2_{L^2(Q)}).
\end{equation}

Similarly, let \( \psi \) be the solution of (3.20) with \( F = y_h - y^{n_h} \) and \( \psi_1 = \pi_h^n \psi. \) Then, by Lemma 3.4, (4.3), (2.14), and the Green's formula,

\[ \|y_h - y^{n_h}\|^2_{L^2(Q)} = \int_0^T (y_h - y^{n_h}, F) \, dt = \int_0^T (y_h - y^{n_h}, \frac{\partial \psi}{\partial t} - \text{div}(A^* \psi)) \, dt \]

\[ = \int_0^T \left( \frac{\partial y_h}{\partial t} (y_h - y^{n_h}, \psi) + a(y_h - y^{n_h}, \psi) \right) dt + (y_h(0) - y^{n_h}(0), \psi(0)) \]

\[ - \int_0^T \left( \frac{\partial y_h}{\partial t} - f - Bu_h, \psi \right) dt + (y_h(0) - y_0, \psi(0)) \]

\[ + \int_0^T (\frac{\partial y_h}{\partial t} - f - Bu_h - \text{div}(A\nabla y_h), \psi - \psi_1) dt + \sum_{t \in \partial T^h} [A\nabla y_h] \cdot \mathbf{n}_x |(\psi - \psi_1) \]

\[ := J_1 + J_2 + J_3. \]

Let

\[ r_y := \frac{\partial y_h}{\partial t} - f - Bu_h - \text{div}(A\nabla y_h). \]

Then, as in (4.16) and (4.18),

\[ J_1 = \int_0^T (r_y, (\pi_h^n - \mathbf{n}) \psi) \, dt \leq C \sum_{t \in \partial T^h} h_t^2 \|r_y\|^2_{0,t} + \delta \|\psi\|^2_{H^1(Q)} \]

\[ \leq C(\bar{\eta}_2^2 + \|y_h - y^{n_h}\|^2_{L^2(Q)}), \]

\[ J_2 = \sum_{t \in \partial T^h} [(A\nabla y_h) \cdot \mathbf{n}_x |(\psi - \psi_1) \leq \sum_{t \in \partial T^h} \|[(A\nabla y_h) \cdot \mathbf{n}_x |\|_{0,t} \|\psi - \psi_1|_{0,t} \]

\[ \leq C \sum_{t \in \partial T^h} h_t \|((A\nabla y_h) \cdot \mathbf{n}_x |\|_{0,t} + \delta \|\psi\|^2_{H^1(Q)} \]

\[ \leq C(\bar{\eta}_3^2 + \|y_h - y^{n_h}\|^2_{L^2(Q)}), \]

and

\[ J_3 = (y_h(0) - y_0, \psi(0)) \leq C \sum_{t \in \{t^k : 0 < t < T\}} h_t \|y_h - y_0\|^2_{0,t} + \delta \|\psi\|^2_{H^1(Q)} \]

\[ \leq C(\bar{\eta}_3^2 + \|y_h - y^{n_h}\|^2_{L^2(Q)}). \]

Hence

\begin{equation}
\|y_h - y^{n_h}\|^2_{L^2(Q)} \leq C(\bar{\eta}_2^2 + \bar{\eta}_3^2 + \bar{\eta}_3^2).
\end{equation}

We complete the proof by combining the estimates (4.20) and (4.21). □

From Lemma 4.1 and Lemma 4.4, we have the following \emph{a posteriori} error estimate.
Theorem 4.1. Let \((y, p, u)\) and \((y_h, p_h, u_h)\) be the solutions of (2.2) and (2.14), respectively. Assume that the conditions in Lemmas 4.1-4.4 are valid, then

\[
\|u - u_h\|_{L^2(Q)}^2 + \|y - y_h\|_{L^2(Q)}^2 + \|p - p_h\|_{L^2(Q)}^2 \leq C \sum_{i=1}^{6} \tilde{\eta}_i^2
\]

where \(\tilde{\eta}_i^2\) are defined in Lemma 4.1 and Lemma 4.4.

Proof. We obtain from (4.2), (4.21), and (4.20) that

\[
\|u - u_h\|_{L^2(Q)}^2 + \|y^{\text{sh}} - y_h\|_{L^2(Q)}^2 + \|p^{\text{sh}} - p_h\|_{L^2(Q)}^2 \leq C \sum_{i=1}^{6} \tilde{\eta}_i^2.
\]

Then the desired results follow from the triangle inequality and

\[
\|p - p^{\text{sh}}\|_{L^2(Q)} \leq C\|y - y^{\text{sh}}\|_{L^2(Q)} \leq C\|u - u_h\|_{L^2(Q)},
\]

which can be derived from (4.5) and (4.6). \(\square\)

5. A PRECONDITIONED PROJECTION ALGORITHM

With the error estimators derived in the above sections, we can then generate adaptive meshes and discretise the control problems into some finite dimensional optimization problems, which can be solved via mathematical programming. There has been extensive study in this area, and many fast numerical algorithms have been developed in the literature, although it is too large to be reviewed here even very briefly. Some of the recent progress in the area has been summarized in [8], [19], and [42]. For instance, recently it was shown in [6], [7], [18] and [42] that the semi-smooth Newton with an active-set strategy can efficiently solve constrained quadratic optimal control problems. Recent studies (see [17]) have recognized the importance of using different meshes for the control and the states in adaptive finite element schemes. We shall adopt this strategy in our numerical tests to save substantial computational work. In the case of multi-set meshes, we used a simple and yet fast projection algorithm, which does not involve identifying and updating active-sets. It works very well with the adaptive multi-mesh discretisation, and can be used to solve large scale control problems. We first state this projection method for an abstract minimization problem. Let \(U\) be a real Hilbert space with \(U' = U\), and \(K\) be a closed convex subset of \(U\). Here we have slightly extended the original meanings of \(U\) and \(K\). However this should not cause any confusion. Consider

\[
\min_{u \in K} J(u)
\]

where \(J(u)\) is a convex functional on \(U\). The optimality condition of (5.1) reads

\[
(J'(u), v - u) \geq 0 \quad \forall v \in K,
\]

where \(J'\) is the \(G\)-differential of \(J\).

Let \(b(\cdot, \cdot)\) be a symmetric and positive definite bilinear form such that there exist constant \(c_0\) and \(c_1\) satisfying

\[
|b(u, v)| \leq c_0 \|u\|_{U} \|v\|_{U}, \forall u, v \in U,
\]

\[
b(u, u) \geq c_0 \|u\|_{U}^2.
\]

Define operator \(b : U \rightarrow U\) by

\[
b(u, v) = b(u, v) \quad \forall u, v \in U.
\]
It is clear that the norm \( \| \cdot \|_b = \sqrt{\langle \cdot , \cdot \rangle} \) is equivalent to the norm \( \| \cdot \|_U \) by the assumption.

Now define the projection operator \( P_K^h : U \rightarrow K \): For given \( w \in U \) find \( P_K^h w \in K \) such that

\[
(5.6) \quad b(P_K^h w - w, P_K^h w - w) = \min_{u \in K} b(u - w, u - w),
\]

which is equivalent to

\[
(5.7) \quad b(P_K^h w - w, v - P_K^h w) \geq 0 \quad \forall v \in K.
\]

It is clear that \( P_K^h \) is well-defined for any closed convex subset in \( U \).

For the minimization problem (5.1), define an iterative scheme \((n = 0, 1, 2, \ldots)\)

\[
(5.8) \quad \begin{cases} 
 b(u_{n+\frac{1}{2}}, v) = b(u_n, v) - \rho_n (J'(u_n), v) & \forall v \in U \\
 u_{n+1} = P_K^h \left( u_{n+\frac{1}{2}} \right) 
\end{cases}
\]

One can just use a fixed step size, or variable ones given by a linear search procedure. It is not difficult to show the following convergence result:

**Theorem 5.1.** Assume that \( J' \) is Lipschitz and uniformly monotone in the sense that there are positive constants \( c, C \) such that

\[
|J'(u) - J'(v)| \leq C \|u - v\|_U, \quad \forall u, v \in U,
\]

\[
(J'(u) - J'(v), u - v) \geq c \|u - v\|_U^2, \quad \forall u, v \in U.
\]

Then there are \( 0 < \delta < 1, \epsilon > 0 \) such that

\[
\|u - u_n\|_b \leq \delta^n \|u - u_0\|_b, \quad n = 0, 1, 2, \ldots,
\]

provided \( \rho_n < \epsilon \).

Let us explain why this simple algorithm of gradient type may work well. Let \( U = U^h, U_n \) be the coordinates of \( u_n \) in \( R^n \), \( b(u, v) = (u, v)_{U^h} \), and \( DJ(U_n) \) be the gradient of \( J(U_n) \). Then the matrix form of the first part of the above algorithm reads:

\[
U_{n+\frac{1}{2}} = U_n - \rho_n M_h^{-1} DJ(U_n),
\]

where \( M_h \) is the mass matrix of \((\cdot, \cdot)_{U^h}\), the preconditioner. It behaves quite differently from the (un-preconditioned) gradient method:

\[
U_{n+\frac{1}{2}} = U_n - \rho_n DJ(U_n),
\]

which may be very slow when \( h \) is small, although now the projection may be very easy to compute, at least for the piecewise constant or linear elements. We then apply the above algorithm to a reduced discretised control problem. For instance, for the new approximation scheme, the reduced control problem reads:

\[
(5.9) \quad \min_{u_h \in U_h} \left\{ \int_0^T \left( g(y_h(u_h)) + h(u_h) \right) dt \right\},
\]

where \( y_h(u_h) \) is the solution of the discretised state equation. Let \( U = X^h \) and \( b(\cdot, \cdot) \) be a bilinear form described as above. An application of (5.8) to the above discretised control problem yields the
following algorithm (PPGA).

\[
\begin{align*}
    b(u_{n+\frac{1}{2}}, v) &= b(u_n, v) - \rho_n \int_0^T (h'(u_n) + B^* p_n, v) \, dt, \quad u_{n+\frac{1}{2}}, u_n \in X^h, \forall v \in X^h \\
    \frac{1}{T} \int_0^T \left( \frac{\partial y_n}{\partial t}, w \right) + a(y_n, w)) \, dt + (y_n(0) - y_0, w(0)) &= \int_0^T (f + B u_n, w) \, dt, \quad \forall w \in W^h, \\
    \frac{1}{T} \int_0^T (-\frac{\partial p_n}{\partial t}, q) + a(q, p_n)) \, dt + (p_n(T), q(T)) &= \int_0^T (g(y_n), q) \, dt, \quad \forall q \in W^h, \\
    u_{n+1} &= P_K^h \left( u_{n+\frac{1}{2}} \right),
\end{align*}
\]

where we have omitted the subscript \( h \). The main computational effort is to solve the two state equations, and to compute the projection \( P_K^h u_{n+\frac{1}{2}} \). In this paper we use a fast algebraic multigrid solver to solve the state equations. Then the key to saving computing time is how to compute \( P_K^h u_{n+\frac{1}{2}} \) efficiently. If one uses the \( C^1 \) finite elements to approximate the control, then one has to solve a global variational inequality, via, e.g., Semi-smooth Newton method. The computational load is not trivial. Our discontinuous discretization makes it possible to explicitly compute \( P_K^h \). For instance, if one uses the piecewise constant elements, \( K^h = \{ u_h : u_h \geq 0 \} \) and \( b(u, v) = (u, v)_U, U = X^h \), then

\[
P_K^h p_h|_{tv} = \max(0, \text{avg}(p_h)|_{tv}),
\]

where \( \text{avg}(p_h)|_{tv} \) is the integral average of \( p_h \) over \( tv \). The solution formulas for the other important cases can be found in [17].

With these ingredients, the algorithm (PPGA) is efficient in solving the control problem \((OCP - OPT)^h\), even with a fixed step size \( \rho \).

6. Numerical Experiments

In this section, we carry out some numerical experiments to demonstrate our results. In most control problems, the optimal control is often of prime interest, and yet the regularity of the control is often the worst. Thus it is important to develop mesh refinement schemes which are most efficient to reduce the error \( \|u - u_h\| \). In practice, there are four major types of adaptive finite element methods, namely, the \( h \)-method (mesh refinement), the \( p \)-method (order enrichment), the \( r \)-method (mesh redistribution), and the \( h-p \) method, which is the combination of \( h \)-method and \( p \)-method. A posteriori error estimates can be used as error indicators to guide the mesh refinement in adaptive finite element methods. Since we wish to implement multi-mesh adaptive schemes, the best choice, and at the same time probably the unique choice, is then the \( h \)-method, which will be briefly explained below. The general idea is to refine the meshes such that the error estimators are equally distributed over the computational mesh. Assume that an \( a posteriori \) error estimator \( \eta \) has the form \( \eta^2 = \sum e_i^2 \), where \( e_i \) is a finite element. In mesh refinement method, at each iteration an average quantity of \( \{ \eta^2_i \} \) is calculated, and each \( \eta^2_i \) is then compared with this quantity. The element \( e_i \) is to be refined if \( \eta^2_i \) is larger than this quantity. As \( \eta^2_i \) reflects the distribution of the total approximation error over \( e_i \), this strategy guarantees that a higher density of nodes is distributed over the area where the error is larger.

Very often the optimal control \( u \) may have gradient jumps along the free boundary of the level set of the co-state \( p = 0 \), though the state or co-state usually is quite regular, see [25]. Therefore the locations of computational difficulties of the control and of the state are in general very different in constrained control problems. Thus different meshes should be used for the control and the states in order to produce most efficient computational meshes. These meshes will be adapted according
to different indicators independently. Our software package AFEPack can help us implement such operations conveniently between different meshes, see [17] for more details. Using different meshes for the control and the state allows very coarse meshes to be used in solving the state and co-state equations. Thus much computational work can be saved since one of the major computational loads is to solve the state and co-state equations repeatedly. This can be clearly seen from our previous numerical experiments conducted in [17].

For the second scheme, we apply the standard mesh refinement technique mentioned above on the whole space-time domain. While for the first scheme, the temporal direction is divided uniformly and we didn’t apply any adaptation in this direction, since it is clear that adaptive refinement in temporal direction for the back-Euler scheme will only lead to uniform like meshes for our examples. For the spatial variables, we apply the mesh adaptation independently on every time step. We obtain different meshes on different time steps, and those meshes are adapted according to their own indicators. We shall use \( \eta_h \) (\( \tilde{\eta}_h \)) as the control mesh refinement indicator, and \( \eta_2 + \eta_3 \) (\( \tilde{\eta}_2 + \tilde{\eta}_3 \)) as the state’s and co-state’s for the backward Euler scheme (the global scheme). The details of the implementation can be found in [17] and [25]. In all our experiments, the numerical examples are the following type of optimal control problems:

\[
\begin{align*}
\min \int_0^1 (g(y) + h(u)) dt \\
\text{s.t. } \frac{\partial y}{\partial t} - \Delta y + \phi(y) = u + f, \quad u \geq 0.
\end{align*}
\]

in which

\[
\begin{align*}
g(y) &= \int_\Omega (y - y_0)^2 \, dx \\
h(u) &= \int_\Omega (u - u_0)^2 \, dx
\end{align*}
\]

and the spatial domain \( \Omega \) here is set as the unit square \([0, 1] \times [0, 1] \). Let \( \Omega^h \) and \( \Omega^h_U \) be partitioned into \( T^h \) and \( T^h_U \) as described Section 2. Piecewise linear finite elements have been used for both the schemes and in all the examples. We may use different meshes for the approximation of the state and the control.

In solving the above optimization problem, we use the preconditional projection gradient method (PPGA) developed in Section 5 with \( b(u, v) = (u, v)_U \).

**Example 1.** For the first example, the data are as follows:
\[
\begin{align*}
\mu(x) &= \sin \pi x_1 \sin \pi x_2 \\
\nu(t) &= \sin \pi t \\
p(x, t) &= \mu(x)\nu(t) \\
z(x, t) &= \begin{cases} 1/2, & \text{if } x_1 + x_2 > 1.0 \\ 0, & \text{otherwise} \end{cases} \\
u_0(x, t) &= 1 - \sin \frac{x_1}{2} - \sin \frac{x_2}{2} + z \\
u(x, t) &= \max(u_0 - p, 0) \\
y_0(x, t) &= 0 \\
y(x, t) &= -\frac{\partial p}{\partial t} - \Delta p + y_0 \\
f(x, t) &= \frac{\partial y}{\partial t} - \Delta y - u \\
\phi(s) &= 0.
\end{align*}
\]

(6.2)

The optimal control has a strong jump (discontinuity), introduced by \( u_0 \). Although the jump does not "move" with time, it may expand or contract with time because the zero-set of the adjoint state \( p \) varies with time. In Table (1) numerical results are presented for the first scheme with 41 time steps, while results for the second scheme are in Table (2). It can be seen that for both schemes, adaptive meshes can substantially save computational work, since the main computing work required in the projection algorithm is to solve the state and co-state equations repeatedly. The projection of the control can be computed via an explicit formula, as seen in [17]. This reconfirms the findings in the studies for the static optimal control in [17] and [25].

It is more interesting to compare the two schemes. It is clear that the global scheme can achieve better approximation using much fewer \( y, p \) dofs. However, the linear system from the backward Euler scheme can be solved via time-match, while this is impossible for the linear system from the global scheme. We have used multi-grid solvers with linear complexity for both the systems. It was found that for the same total dofs the backward Euler system can be solved faster, though it yields less accuracy. To increase the approximation accuracy, one has to further refine the space grids. This leads to a larger backward Euler linear system so that both the schemes have a similar computational load for this example. This is mainly because the jump only expand or contract, and thus is relatively simpler in the \( x-t \) space. It is clear that if the optimal control is very smooth, then the backward Euler scheme is more efficient than the global scheme. It is also clear that in general higher order time-stepping schemes have the same nature as the backward Euler scheme, and thus will not catch the complex control singularity in \( x-t \) space well.

**Example 2.** In this example, the location of the singularity of the optimal control moves with time. The purpose is two-fold. Firstly we wish to see if the posteriori error estimators and the mesh adaptation can catch a moving discontinuity. Secondly we wish to compare the performance of the two schemes in such a case. The data are as follows, and the numerical results are presented in the
Following two tables.

\[
\begin{align*}
\mu(x) &= \sin \pi x_1 \sin \pi x_2 \\
\nu(t) &= \sin \pi t \\
p(x, t) &= \mu(x) \nu(t) \\
z(x, t) &= \begin{cases} 
1/2, & \text{if } \psi(x, t) \geq 0 \\
0, & \text{otherwise} 
\end{cases} \\
\psi(x, t) &= \sin 2\pi(t + 1/2) \{ \cos \pi(t - 3/4)(x_1 - 1/2) \\
&\quad + \sin \pi(t - 3/4)(x_2 - 1/2) \} 
\end{align*}
\]

\[ (6.3) \]

| mesh | # nodes | # sides | # elements | # dofs | \[ \sum_i |u(t_i) - u_h(t_i)|_L^2 \] | Total L\textsuperscript{2} error |
|------|---------|----------|------------|--------|-------------------------------|-------------------------------|
|      |         |          |            |        |                               |                               |
| on uniform mesh | 82600 | 84665 | 84665 | 28718 | 1.53016e-02 | 1.59392e-02 |
| on adaptive mesh | 16678 | 16678 | 5.17266e-02 | 6.66792e-02 | 5.15417e-02 |

| mesh | # nodes | # edges | # faces | # elements | # dofs | \[ \sum_i |u(t_i) - u_h(t_i)|_L^2 \] | Total L\textsuperscript{2} error |
|------|---------|---------|---------|------------|--------|-------------------------------|-------------------------------|
|      |         |         |         |            |        |                               |                               |
| on uniform mesh | 56853 | 383764 | 646912 | 320000 | 1280000 | 29633 | 9339 | 9339 |
| on adaptive mesh | 56853 | 383764 | 646912 | 320000 | 1280000 | 176988 | 59472 | 59472 |

Table 1. Example 1 via backward Euler with time step \( \Delta t \).
Some numerical results are presented in Table (3) and Table (4). It can be seen from the tables that in this example, a similar behaviors of the two schemes have been observed. This time the backward Euler scheme gives much worse approximation results, while the global scheme did not change much. It is also clear that for the backward Euler scheme one has to refine the time-meshes to reduce approximation errors, since the total $L^2$ error in the space variables at each time step is already of higher order comparing with the total approximation $L^2$ error. The main disadvantage of the backward Euler scheme is that it is impossible to locally refine in $x$-$t$ space. In fact, one has to uniformly refine the time-grids if the discontinuity of the optimal control crosses the $x$-$t$ space as in this example. Then the scheme loses much its computational efficiency. This will be more clearly seen in the next example.

**Example 3.** In this example, the optimal control has a similar nature to that in Example 2, although the state equation is replaced by the following nonlinear equation:

$$
\frac{\partial u}{\partial t} - \Delta y + y^3 = u + f
$$

and the dual equation now is

$$
-\frac{\partial p}{\partial t} - \Delta p + 3y^2p = y - y_0.
$$

The problem is still well-posed with such a special nonlinear term. The data under testing are as follows:
<table>
<thead>
<tr>
<th>mesh info</th>
<th>on uniform mesh</th>
<th>on adaptive mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td># nodes</td>
<td>82600</td>
<td>93339</td>
</tr>
<tr>
<td># sides</td>
<td>241280</td>
<td>263569</td>
</tr>
<tr>
<td># elements</td>
<td>158720</td>
<td>170270</td>
</tr>
<tr>
<td># dofs</td>
<td>476160</td>
<td>510810</td>
</tr>
<tr>
<td>$\sum_u</td>
<td>u(t_i) - u_h(t_i)</td>
<td>_{L^2}$</td>
</tr>
<tr>
<td>Total $L^2$ error</td>
<td>8.97625e-01</td>
<td>8.95672e-01</td>
</tr>
</tbody>
</table>

Table 5. Example 3 via backward Euler with time step 41

<table>
<thead>
<tr>
<th>mesh info</th>
<th>on uniform mesh</th>
<th>on adaptive mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td># nodes</td>
<td>165200</td>
<td>188219</td>
</tr>
<tr>
<td># sides</td>
<td>482560</td>
<td>531166</td>
</tr>
<tr>
<td># elements</td>
<td>317440</td>
<td>343027</td>
</tr>
<tr>
<td># dofs</td>
<td>952320</td>
<td>1029081</td>
</tr>
<tr>
<td>$\sum_u</td>
<td>u(t_i) - u_h(t_i)</td>
<td>_{L^2}$</td>
</tr>
<tr>
<td>Total $L^2$ error</td>
<td>4.52767e-01</td>
<td>4.48856e-01</td>
</tr>
</tbody>
</table>

Table 6. Example 3 via backward Euler with time step 81

\[
\begin{align*}
\mu(x) &= \sin 2\pi x_1 \sin 2\pi x_2 \\
\nu(t) &= \sin 2\pi t \\
p(x, t) &= 0 \\
u_0(x, t) &= \max(4\pi^2 \mu(x) \nu(t), 0) \\
u(x, t) &= u_0 \\
y_0(x, t) &= \mu(x) \nu(t) \\
y(x, t) &= y_0 \\
f(x, t) &= \frac{\partial y}{\partial t} - \Delta y + \phi(y) - u \\
\phi(s) &= s^3.
\end{align*}
\]

Some numerical results are presented in Table (5) - Table (7). Again for the backward Euler scheme one has to uniformly refine the time-meshes to reduce approximation errors, since the total $L^2$ error in the space variables at each time step is already of higher order comparing with the total approximation $L^2$ error. Indeed we doubled the time grids, and reduced the total error by a half. However it is clear that to achieve the same error tolerance as the global scheme, the total dofs have to be much more than those of the global scheme. In fact, they will be of order difference. Thus the global scheme is much more efficient than the backward Euler scheme in this example.
<table>
<thead>
<tr>
<th>mesh</th>
<th>on uniform mesh</th>
<th>on adaptive mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>u</td>
<td>y</td>
</tr>
<tr>
<td># nodes</td>
<td>56853</td>
<td>56853</td>
</tr>
<tr>
<td># edges</td>
<td>383764</td>
<td>383764</td>
</tr>
<tr>
<td># faces</td>
<td>646912</td>
<td>646912</td>
</tr>
<tr>
<td># elements</td>
<td>320000</td>
<td>320000</td>
</tr>
<tr>
<td># dofs</td>
<td>1280000</td>
<td>56853</td>
</tr>
<tr>
<td>Total $L^2$ error</td>
<td>$6.69603e-02$</td>
<td>$4.79429e-03$</td>
</tr>
</tbody>
</table>

Table 7. Example 3 via global discretization

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